Lecture 2

Probability

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Basic Probability Concepts

- Foundation of statistics

because of the concept of sampling and the concept of variation or dispersion and how likely an observed difference is due to chance

- Probability statements used frequently in statistics
  - e.g., we say that we are 90% sure that an observed treatment effect in a study is real

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Characteristics of Probabilities

- Probabilities are expressed as fractions between 0.0 and 1.0
  - e.g., 0.01, 0.05, 0.10, 0.50, 0.80
  - Probability of a certain event = 1.0
  - Probability of an impossible event = 0.0
- Application to biomedical research
  - e.g., ask if results of study or experiment could be due to chance alone
  - e.g., significance level and power

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Definition of Probabilities

- If some process is repeated a large number of times, **n**, and if some resulting event with the characteristic of **A** occurs **m** times, the relative frequency of occurrence of **A**, **m/n**, will be approximately equal to the probability of **A**:

$$P(A)=m/n$$

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Some Properties of Probabilities

1. Probability of an event is a non-negative number
   - Given some process (or experiment) with n mutually exclusive outcomes (events), $E_1, E_2, \ldots, E_n$, the probability of any event E_i is assigned a nonnegative number

     $$P(E_i) \geq 0$$

   - key concept is mutually exclusive outcomes - cannot occur simultaneously

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2. Sum of the probabilities of mutually exclusive outcomes is equal to 1
   - Property of exhaustiveness
     - refers to the fact that the observer of the process must allow for all possible outcomes
   - $P(E_1) + P(E_2) + \ldots + P(E_n) = 1$
   - key concept is still mutually exclusive outcomes

3. Probability of occurrence of either of two mutually exclusive events is equal to :

   $P(A \cup B)=P(A)+P(B)$
   - If A and B are two mutually exclusive events.

   $P(A \cup B) = P(A) + P(B)-P(A \cap B)$
   - If they are not mutually exclusive.

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4. For two *independent events*, A and B, occurrence of event A has no effect on probability of event B
   - $P(A \cup B) = P(A) + P(B)-P(A \cap B)$
   - $P(A/B) = P(A)$
   - $P(B/A) = P(B)$
   - $P(A \cap B) = P(A) \times P(B)$
5. Conditional probability
   - Conditional probability of B given A is given by:
   - $P(B/A) = P(A \cap B) / P(A)$
   - Probability of the occurrence of event B given that event A has already occurred.
   - Ex. given that a test for disease is positive, what is the probability that the patient has this disease?

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6. Multiplicative Law
   - For any two events A and B,
   - $P(A \cap B) = P(A)\ P(B/A)$
     - Joint probability of A and B = Probability of B times Probability of A given B
7. Addition Law
   - For any two events A and B
   - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
     - Probability of A or B = Probability of A plus Probability of B minus the joint Probability of A and B

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Probability distributions of discrete variables

- A **table, graph**, or other device used to specify all possible values of a discrete random variable along with their respective probabilities
  - $P(X=x_i)$
- **Tables** – value, frequency, probability
- **Graph** – usually bar chart or histogram

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Cumulative Distributions

- Probability that X is less than or equal to a specified value, x_i
- Calculated by adding successive probabilities $P(X=x_i)$
- Easier to work with for many applications
- $P(X \le x_i)$
- Theoretical distribution can be compared to sample distribution to determine appropriateness of theoretical distribution

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Theoretical Probability Distributions

- Isn't observation enough?
  - If we know that data are from a certain distribution, then we know a lot about it such as:
    - Means, standard deviations, other measures of dispersion
  - That knowledge makes it easier to make statistical inference; i.e., to test differences
- There are several standard statistical distributions such as:
  - Binomial
  - Poisson
  - Normal

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Binomial Distribution

- Derived from a series of binary outcomes called a Bernoulli trial
- A sequence of Bernoulli trials forms a Bernoulli process under the following conditions
  - Each trial results in one of two possible, mutually exclusive, outcomes: "success" and "failure"
  - Probability of success, p, remains constant from trial to trial. Probability of failure is $q = 1-p$.
  - Trials are independent; that is, success in one trial does not influence the probability of success in a subsequent trial.

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Example

- Probability of a certain sequence of binary outcomes (Bernoulli trials) is a function of *p* and *q*.
- For example, a particular sequence of 2 "successes" and 3 "failures" can be represented by p*p*q*q*q; = p^2q^3
- However, if we ask for the probability of 2 "successes" and 3 "failures" in 10 trials, then we need to know how may possible combinations of 2 successes and 3 failures out of all of the possible outcomes there are.

Binomial Distribution

- The binomial probability density function is defined as following:
  - $f(x) = C_x^n p^x q^{n-x}$ for x=0,1,2,3…,n
    = 0 elsewhere
- Mean
  - $\mu = np$
- Variance
  - $\sigma^2 = np(1-p)$
- Appropriateness in sampling situations
  - Appropriate if n small relative to N
  - Otherwise, not really in a sampling situation

Poisson Distribution

- If x is the number of occurrences of some random event in an interval of time or space, the probability that x will occur is given by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

  - where λ is the average number of occurrences of the random event in the interval t.
  - e = 2.7183
- Parameters of the Poisson distribution
  - Mean = Variance = λ

Poisson Process

- Assumptions
  - Occurrences of events are independent.
  - Theoretically, an infinite number of occurrences of the event must be possible in the interval
  - Probability of the single occurrence of the event in a given interval is proportional to the length of the interval.
  - In an infinitesimally small portion of the interval, the probability of more than one occurrence of the event in negligible; i.e., the event times are unique and discrete

Poisson Approximation to the Binomial Distribution

- When n is large and p is small, the Poisson is a reasonable approximation to the Binomial
  - Poisson is easier to work with
  - B(x,n,p) → P(λ) where λ=np