Confidence Interval
or
Interval Estimation

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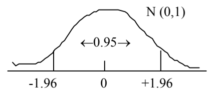
- From the statistical tables for a Standard Normal distribution, we note that

Area Under Density Function	From	To
0.90	-1.64	1.64
0.95	-1.96	1.96
0.99	-2.58	2.58



- From the **central limit theorem**, if $\bar{x}$ and s^2 are mean and variance of a random sample of size n, $n >$ (from a large parent population), then we can say that 90% C.I. for population mean μ

$$P\{-1.64 \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le +1.64\} \approx 0.90$$

where s replaces σ, if σ is unknown

- or $P\{\bar{x}-1.64\sigma/n \le \mu \le \bar{x}+1.64\sigma/n\} \approx 0.90$

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Confidence Interval for the mean

1. If sample size is less than 30 (n < 30) and population variance unknown

$$\bar{X} \pm t_{(n-1,\alpha/2)}\, s/\sqrt{n}$$

Thus, for a 95% confidence interval for a sample with 20 observations,

$$\bar{X} \pm 2.093\; s/\sqrt{n}$$

- 2.093 is the $\mathbf{t_{(19,\ 0.025)}}$ percentile point.

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Confidence Interval for the mean- contd.

2. A normal distribution with known population variance or large sample size (n > 30)

$$\bar{X} \pm Z_{\alpha/2}\, s/\sqrt{n}$$

Thus, for a 95% confidence interval for a sample with 50 observations,

$$\bar{X} \pm 1.96\, s/\sqrt{n}$$

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Confidence Interval for Variance

- A normal distribution
  - Depends on the chi-square distribution, which is the distribution of squared normal variables
  - So, if underlying distribution is not normal, the estimation will be poor

$$(n-1)s^2 / \chi^2_{n-1,\ \alpha/2},\ (n-1)s^2 / \chi^2_{n-1,\ \alpha/2}$$

  - Not symmetric
- Rarely used in practice as such, but equality of variances are tested (using the F-distribution) in a number of procedures

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CI for Difference between Two Population Means

Two random samples are drawn from two populations

1. If population variances are known, C. I is given by

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

- Use sample variance for sample means
- When $n_1+n_2-2 < 30$, use t distribution

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CI for Difference between Two Population Means

- Population Variances are ***Equal*** and ***Unknown***
  - Can use pooled estimate of variance as on previous slide except that a pooled estimate of the sample variance is used

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

$$(\overline{x}_1 - \overline{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

- Population Variances are ***Unequal*** and ***Unknown***

$$(\overline{x}_1 - \overline{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

CI for Population Proportions

- Same procedures as with means, **p** for $\overline{X}$ and **p(1-p)** for s^2
- For example,

$$\hat{p} \pm z_{(\alpha/2)}\sqrt{\hat{p}(1-\hat{p})/n}$$

- Is the 100(1-α) percent confidence interval for p
- Similarly for difference between two population proportions

Example**:** confidence intervals

- **Attribute Sampling**

  A random sample of size, n = 25 has $\overline{x}$ = 15 and s = 2.

  Then a 95% confidence interval for m is

$$P\{\bar{x} - 1.96\, {s}/{\sqrt{n}} \le \mu \le \bar{x} + 1.96\, {s}/{\sqrt{n}}\} \approx 0.95$$

  i.e. 15 + 1.96 (2 / 5) so, C.I. is 14.22 to 15.78

- **Proportionate Sampling**

  A random sample of size n = 1000 has p = 0.40

$$P\{p - 1.96\sqrt{p(1-p)/n} \le P \le p + 1.96\sqrt{p(1-p)/n}\} \approx 0.95$$

  A 95% confidence interval for P is 0.40 ±0.03 (i.e.) 0.37 to 0.43.

Examples:

- A random sample of size n = 10, drawn from a large parent population, has a mean of 12 and a standard deviation s = 2. Then a **99%** confidence interval for the **parent** mean is

$$\bar{x} \pm 3.25\, {s}/{\sqrt{n}}$$

  ie. 12 + 3.25 (2)/3 that is an interval 9.83 to 14.17
- and **95%** confidence limits for the **parent** mean is

$$\bar{x} \pm 2.262\, {s}/{\sqrt{n}}$$

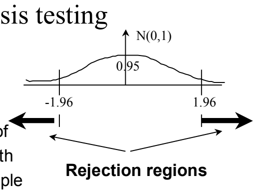
  ie 12 + 2.262 (2)/3 that is an interval 10.492 to 13.508.

- Note that for n = 1000, $1.96\sqrt{p(1-p)/n} \approx 0.03$ for values of p between 0.3 and 0.7.

  Refer to 3% "swing" or "inherent error"

Hypothesis Testing

Exmple: Hypothesis testing



- Suppose that it is claimed that the average survival time of patients with cancer at a specific site = 60 months. A random sample of n= 49 patents gives a mean of 55 with a standard deviation of 2. Is the sample finding **consistent** with the claim?

  We regard the original claim as a **null hypothesis (H_0)** which is tentatively accepted as **true**: $H_0 : \mu = 60,$ with $H_1 : \mu \neq 60$

  If H_0 **true**, **test statistic** $t_{n-1} = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ as above

Testing a Single Mean – One Sided

- Test to compare the mean of a normal distribution against a pre-specified value, such as a population mean
- Test statistic is

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

- for H_0: $\mu=\mu_0$ vs. H_1: $\mu<\mu_0$ or $\mu>\mu_0$
  - with σ unknown
- Reject if $t < t_{(n-1, \alpha)}$, accept otherwise
  - t is called a test statistic
  - $t(n-1,\alpha)$ is called a critical value
- Alternatively, use p-values directly

Testing a Single Mean – Two Sided

- In some cases, may not be sure of which direction the difference may be going
- In this case, we are testing
  - H_0: $\mu=\mu_0$ vs. H_1: $\mu \neq \mu_0$
  - Test statistic is the same

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

  - Reject if $t < t_{(n-1, \alpha/2)}$, accept otherwise
  - As with confidence intervals, two sided tests have higher critical values

Paired T-Test

- Test to deal with two observations on the same individuals
  - Before vs. after treatment
  - Before vs. after some biological milestone
- Approach is to calculate differences between two measurements for each individual and then test the difference against zero
  - $d_i = X_{i1} – X_{i2}$
  - Test statistic

$$t = \bar{d} \Big/ \left(s_d / \sqrt{n}\right)$$

  - And test against the t-distribution with d.f. = n-1 and associated p-value

Testing a Single Variance

- Test statistic **(Chi-Square)** is:

$$\chi^2 = (n-1)\, s^2/\sigma_0^2$$

- One sided test, H_0: $\sigma^2= \sigma_0^2$ vs. H_1: $\sigma^2 < \sigma_0^2$ or $\sigma^2 > \sigma_0^2$
- Reject if $\chi^2 < \chi^2_{(n-1, \alpha)}$, accept otherwise
- Two sided test, H_0: $\sigma^2= \sigma_0^2$ vs. H_1: $\sigma^2 \neq \sigma_0^2$
- Reject if $\chi^2 < \chi^2_{(n-1, \alpha/2)}$, accept otherwise

Testing a Population Proportion

- Extension of the approaches for mean.

Test statistic is given by

$$z = (\hat{p} - p_0) \Big/ \sqrt{p_0 q_0 / n}$$

1. One sided, H_0: $P=P_0$ and H_1: $P<P_0$ or $P>P_0$
   compare Z with Z_α
2. Two sided, H_0: $P=P_0$ and H_1: $P\neq P_0$
   compare Z with $Z_{\alpha/2}$

Comparison of Means of Two independent Populations

- Assume now that samples are independent
- We test H_0: $\mu_1= \mu_2$ vs. H_1: $\mu_1 \neq \mu_2$

1. If the populations variances are ***Unknown*** and E***qual***

- Test statistic

$$t = (\bar{x}_1 - \bar{x}_2) / s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Where s_p is called a pooled estimate of the standard deviation and is given as

$$s_p = \sqrt{\frac{\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]}{n_1 + n_2 - 2}}$$

Contd.

2. If the populations variances are ***Unknown*** and **U*nequal***

- Test statistic

$$t = (\bar{x}_1 - \bar{x}_2) / \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- Reject if $t < t_{(n-1, \alpha/2)}$, accept otherwise

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Testing for Equality of Two Variances

- How do we test for equal variances for the t-test
- We calculate the ratio of the variances

$$\mathbf{F = s_1^2 / s_2^2}$$

- Test that against the F distribution with n_1 numerator and n_2 denominator degrees of freedom
  - Also called the F test – a major part of regression analysis
  - Two-sided tests, so we reject for small and large values of the F statistic
- Because it is two-sided, does not matter which variance is in the numerator vs. the denominator

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Continued

- **TWO-SAMPLE**

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

$$F_{\alpha/2} < \frac{s_1^2 / \sigma_1^2}{s_2^2 / \sigma_2^2} < F_{1-(\alpha/2)}$$

after manipulation – gives

$$\frac{s_1^2 / s_2^2}{F_{1-(\alpha/2)}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2 / s_2^2}{F_{\alpha/2}}$$

and where, conveniently:

$$F_{1-\alpha/2, v_1, v_2} = \frac{1}{F_{\alpha/2, v_2, v_1}}$$

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Examples: H.T. for a single proportion

1. In a survey of injecting drug users, 18 out of 423 were HIV positive. Claim fewer than five percent in IDU population HIV positive? Hypothesised proportion = 0.05 gives

$H_0 : p \geq 0.05$ $\quad \hat{p} = 18/423 = 0.0426$

while

$H_1 : p < 0.05$ $\quad \hat{\sigma}_p = \sqrt{(0.05)(0.95)/423}$

1-sided at $\alpha = 0.01$, has a $Z = -2.33$, while from data:

$$Z = \frac{0.0426 - 0.05}{\sqrt{(0.05)(0.95)/423}} = -0.70$$

We accept H_0 at $\alpha = 0.01$. Clearly, test inconclusive at $\alpha = 0.01$

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Examples: H.T. for two proportions

2. Two groups of patients, 55 with hypertension of whom 24 on special diet, 149 without, of whom 36 on special diet. Can we say?

$H_0 : p_H \leq p_{\bar{H}}$ *i.e.* $p_H - p_{\bar{H}} \leq 0$? Test at 5% level of significance.

From data, we have,

$\hat{p}_H = 0.4364, \hat{p}_{\bar{H}} = 0.2416$ and

$\bar{p} = (24 + 36)/(55 + 149) = 0.2941$

$$Z = \frac{(0.4364 - 0.2416)}{\sqrt{\frac{(0.2941)(0.7059)}{55} + \frac{(0.2941)(0.7059)}{149}}} = 2.71$$

As $Z_{0.05}$= 1.65 , then we Reject H_0 at $\alpha=0.05$

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Example: H.T. for a single Variances

- Given a simple random sample, size 12, of animals studied to examine release of mediators in response to allergen inhalation. Known S.E. of sample mean = 0.4 from subject measurement. Can we claim on the basis of data that **population** variance is not 4?

$H_0 : \sigma^2 = 4$ *vs* $H_0 : \sigma^2 \neq 4$

From χ^2_{n-1} tables, critical value $\chi^2_{11, 0.25}$ is 21.920, whereas the data give

$s^2 = 12(0.4)^2 = 1.92$ and

$$\chi_c^2 = \frac{(11)(1.92)}{4} = 5.28$$

So cannot reject H_0 at α=0.05

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Example: H.T. for two Variances

- Two different microscopic methods available. Repeated observations on standard object give estimates of variance:

$$A : n_1 = 11, s_1^2 = 1.232 \quad B : n_2 = 20, s_2^2 = 0.304$$

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

Test statistic

$$F = \frac{s_1^2}{s_2^2} = \frac{1.232}{0.304} = 4.05$$

where critical values for dof 10 and 19 = 2.817 for $\alpha = 0.025$. Reject H_0