SETS, RELATIONS, AND FUNCTIONS

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September 27, 2009

Sets

- A **set** is a collection or group of objects or elements or members (Cantor 1895).
  - the collection of the four letters a, b, c, and d is a set, which we may name L:

$$L = \{a, b, c, d\}$$

  - objects comprising a set $\Rightarrow$ elements or members.
  - b is an element of the set L; b is in L; L contains b:

$$b \in L$$

  - z is not an element of L:

$$z \notin L$$

Sets

- In a set we do **NOT** distinguish repetitions or orders of the elements.
  - $\{red, blue, red\}$ equals to $\{red, blue\}$
  - $\{3, 1, 9\}, \{9, 3, 1\}$, and $\{1, 3, 9\}$ are the same set
- Two sets are equal (that is, the same) **if and only if (iff)** they have the same elements.
- The elements of a set need not be related in any way.
  - $\{3, red, \{d, blue\}\}$ is a set with three elements
- A set may have only one element $\Rightarrow$ a **singleton**.
  - $\{1\}$ is a singleton.
  - $\{1\}$ and 1 are quite different.

Notations

- list all the elements, separated by commas and included in braces:
  - $S = \{a, b, c, d\}$
- brace notation with ellipses: using the three dots and your intuition in place of an infinitely long list:
  - $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
    the natural numbers.
  - $\mathbf{S} = \{\ldots, -3, -2, -1\}$
    the negative integers.
- refer to other sets and to properties that elements may or may not have:
  - if a set **A** has been defined, and **P** is a property that elements of **A** may or may not have, then we can define a new set:
  - $\mathbf{B} = \{x : x \in A \text{ and } x \text{ has property } \mathbf{P}\}$
  - $\mathbf{O} = \{x : x \in \mathbf{N} \text{ and } x \text{ is not divisible by } 2\}$
    the odd natural numbers.

Common Universal Sets

- $\mathbb{R}$ = reals
- $\mathbb{N}$ = natural numbers = $\{0, 1, 2, 3, ...\}$, the *counting* numbers
- $\mathbb{Z}$ = integers = $\{.., -3, -2, -1, 0, 1, 2, 3, 4, ..\}$
- $\mathbb{Q}$ = rationals = m/n, where $m, n \in \mathbb{Z}, n \neq 0$
- $\emptyset$ = *empty* set = *void* set =*null* set = a set with no element at all
  - there can be only one such set.
  - the assertion $x \in \emptyset$ is always false.
- Note already that some sets are *infinite* . . .

Subsets

- A set A is a subset of a set B, if each element of A is also an element of B.
- **Definition**: The set A is a *subset* of the set B, denoted $A \subseteq B$, iff

$$\forall x : x \in A \Rightarrow x \in B$$

- Note:
  - $\emptyset$ is a subset of every set.
  - A set is always a subset of itself.
- **Definition**: If $A \subseteq B$ but $A \neq B$ then we say A is a *proper* subset of B, denoted $A \subset B$.
- Question: How to prove that two sets A and B are equal?
  - $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

Power Sets

- **Definition**: The set of all subsets of a set A, denoted $\wp(A)$, is called the *power set* of A.
- Example: If $A = \{a, b\}$ then:

$$\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- including $\emptyset$ and A itself.

Cardinality

- **Definition**: The number of (distinct) elements in A, denoted $|A|$, is called the *cardinality* of A.
- Example: $A = \{a, b\}$
  - $|\{a, b\}| = 2$
  - $|\wp(\{a, b\})| = 4$
- Useful Fact: $|A| = n$ implies $|\wp(A)| = 2^n$
- If the cardinality is a natural number (in $\mathbb{N}$), then the set is called *finite*, otherwise it is *infinite*.
- Note: A is finite and so is $\wp(A)$.
- $\mathbb{N}$ is infinite since $|\mathbb{N}|$ is not a natural number. It is called a *transfinite cardinal number*.
  - denumerable (countably infinite) sets.

Operations on Sets

- *Union*: $A \cup B \quad = \quad \{x : x \in A \ \vee \ x \in B\}$
  - Example: $A = \{a, b, c\},\ B = \{c, d, e\},\ A \cup B = \{a, b, c, d, e\}$
- *Intersection*: $A \cap B \quad = \quad \{x : x \in A \ \wedge \ x \in B\}$
  - Example: $A = \{a, b, c\},\ B = \{c, d, e\},\ A \cap B = \{c\}$
- *Difference*: $A - B \quad = \quad \{x : x \in A \ \wedge \ x \notin B\}$
  - Example: $A = \{a, b, c\},\ B = \{c, d, e\},\ A - B = \{a, b\}$
- *Symmetric Difference*: $A \Delta B = (A - B) \cup (B - A)$
  - Example: $A = \{a, b, c\},\ B = \{c, d, e\},\ A \Delta B = \{a, b, d, e\}$
- Note: Sets are *disjoint* if $A \cap B = \emptyset$.

Pairs

- How can we express relations between objects in the language of sets?
  - $\{4, 7\}$ is the same thing as $\{7, 4\}$, no order information.
- **Ordered Pair**: $\langle x, y \rangle$ is the *ordered pair* of x and y, denoted (x, y) in some texts.
- Note: $\langle x, y \rangle \neq \{x, y\}$ as the latter is unordered.
  - Example: $\langle 4, 7 \rangle$, $\langle 9, 9 \rangle$, $\langle m, n \rangle$
- **Definition**: The **Cartesian product** of A with B, denoted $A \times B$, is the set of *ordered pairs* $\{<a, b> | a \in A \wedge b \in B\}$.
  - Example: $A = \{a, b\}$, $B = \{1, 2, 3\}$
    $A \times B = \{<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>\}$
- Question 1: What is $B \times A$?
- Question 2: If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

Binary relations

- Any subset R of $A \times B$ is a *binary relation* between A and B. Given a binary relation R, we can associate with it a predicate $R(x, y)$ which is true iff $\langle x, y \rangle \in R$.
  - $\{< 1, 6 >, < 1, c >, < c, d >, < 9, d >\}$ is a binary relation on $\{1, 3, 9\}$ and $\{b, c, d\}$.
  - $\{< i, j >: i, j \in \mathbf{N} \text{ and } i < j\}$ (the less-than relation) is a subset of $\mathbf{N} \times \mathbf{N}$.
- Given a binary relation $R \subseteq A \times B$, the inverse relation of R is defined as $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$. Obviously $(R^{-1})^{-1} = R$.

n-Tuples and *n*-ary relations

- Extending pairs to *n*-tuples is straightforward.
- $\langle x_1 \ldots x_n \rangle$ is a sequence of ordered *n*-tuples.
  - Ordered 2-tuples are the same as the ordered pairs.
  - ordered 3-, 4-, 5-, and 6-tuples $\Rightarrow$ ordered triples, quadruples, quintuples, and sextuples.
- The Cartesian product of n sets $A_1 \ldots A_n$ is defined as $A_1 \times A_2 \times \ldots \times A_n = \{\langle a_1, a_2 \ldots a_n \rangle : (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \ldots \wedge (a_n \in A_n) \rangle$.
- Any subset R of $A_1 \times A_2 \times \ldots \times A_n$ is an *n-ary relation* between the n sets $A_1 \ldots A_n$. Given an *n*-ary relation R, we can associate with it a predicate $R(x_1 \ldots x_n)$ which is true iff $\langle x_1 \ldots x_n \rangle \in R$.
  - 1-, 2-, and 3-ary relations $\Rightarrow$ unary, binary, and ternary relations.

Functions

- A function is an association of each object of one kind with a unique object of another kind.
- A *function* (*mapping*, *map*) f from set A to set B is denoted by $f : A \to B$.
  - f associates with each x in A one and only one y in B. (special type of binary relations)
  - A is called the *domain* and B is called the *codomain*.
  - The *range* of f, denoted by $f(A)$, is the set of all images of points in A under f.
- If $f : A_1 \times A_2 \times \ldots \times A_n \to B$, and $f(a_1, \ldots, a_n) = b$, where $a_i \in A_i$ for $i = 1, \ldots, n$ and $b \in B$:
  - $a_1, \ldots, a_n$ are called **arguments** of f
  - b is called the corresponding **value** of f

Injections, Surjections and Bijections

- Let f be a function from A to B.
- **Definition**: f is *one-to-one* (denoted 1-1) or *injective* if $a \neq b$ implies $f(a) \neq f(b)$.
- **Definition**: f is *onto* or *surjective* if $f(A) = B$.
- **Definition**: f is *bijective* if it is surjective and injective (one-to-one and onto).
- Example: if S is the set of countries in the world, C is the set of cities in the world, C_0 is the set of capital cities in the world, and function $g : S \rightarrow C_x$ is specified by

$$g(s) = \text{the capital of country } s$$

  - g is *one-to-one* from S to C.
  - g is *bijective* between S and C_0.

Inverse functions

- A binary relation R always admits a unique inverse relation R^{-1} while only the injective functions admit an inverse function.
- A function $f : A \longrightarrow B$ may fail to have an inverse if there is some element $b \in B$ such that $f(a) \neq b$ for all $a \in A$.
- **Definition**: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.

Composition

- **Definition**: Let $f : B \rightarrow C, g : A \rightarrow B$. The *composition* of f with g, denoted $f \circ g$, is the function from A to C defined by $f \circ g(x) = f(g(x))$.
  - Example: If f is the function that assigns to each dog its owner and g assigns to each person his or her age, then $f \circ g$ assigns to each dog the age of its owner.

Countable Sets

- How to compare the size(cardinality) of sets that are infinitive?
  - Two sets have the same cardinal number if and only if there is a bijection between them.
- If $|A| = |\mathbb{N}|$, the set A is *countably infinite* or *denumerable*. The (transfinite) cardinal number of the set $\mathbb{N}$ is *aleph null* $= \aleph_0$.
- **Definition**: If a set has the same cardinality as a subset of the natural numbers $\mathbb{N}$, then the set is called *countable*.
  - A set is countable if it is finite or countably infinite.
  - The union of any finite number of countably infinite sets is countably infinite.
  - The union of a countably infinite collection of countably infinite sets is countably infinite.

Uncountable Sets

- If a set is not countable we say it is *uncountable*.
- The following sets are uncountable (we show later):
  - The real numbers in [0, 1]
  - $\wp(\mathbb{N})$, the power set of $\mathbb{N}$
  - The set of functions from $\mathbb{N}$ to $\mathbb{N}$
- Note: With infinite sets proper subsets can have the same cardinality. This cannot happen with finite sets.
- Countability carries with it the implication that there is a *listing* or *enumeration* of the elements of the set.

Examples

- **Theorem**: $|A| \leq |B|$ if there is an injection from A to B
  - **Example**: if A is a subset of B then $|A| \leq |B|$
  - **Proof**: the function $f(x) = x$ is an injection from A to B
- **Theorem**: $|A| = |B|$ iff there is is a bijection from A to B
  - **Example**: $|\mathbb{E}| = |\mathbb{N}|$, where $\mathbb{E}$ is the set of even integers (even though $\mathbb{E}$ is a proper subset of $|\mathbb{N}|$)
  - **Proof**: Let $f(x) = 2x$. Then f is a bijection from $\mathbb{N}$ to $\mathbb{E}$:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & \ldots \end{array}$$