CHOMSKY HIERARCHY

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Avram Noam Chomsky

- Born on December 7, 1928, currently professor emeritus of linguistics at MIT.
- American linguist, philosopher, cognitive scientist, political activist, author, and lecturer.
- One of the fathers of modern linguistics.
- Created the theory of generative grammar, which has undergone numerous revisions and has had a profound influence on linguistics.
- Sparked the cognitive revolution in psychology.

Biography

- From 1945, studied philosophy and linguistics at the University of Pennsylvania.
- PhD in linguistics from University of Pennsylvania in 1955.
- 1956, appointed full Professor at MIT, Department of Linguistics and Philosophy.
- 1966, Ferrari P. Ward Chair; 1976, Institute Professor; currently Professor Emeritus.

Contribution

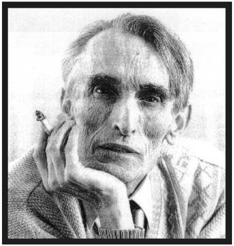
- Linguistics
  - Transformational grammars
  - Generative grammar
  - Language aquisition
- Computer Science
  - Chomsky hierarchy
  - Chomsky Normal Form
  - Context Free Grammars
- Psychology
  - Cognitive Revolution (1959)
  - Universal grammar

Universal grammar

- Noam Chomsky's approach to the study of language is known as universal grammar:
  - the human brain contains a limited set of rules for organizing language. In turn, there is an assumption that all languages have a common structural basis. This set of rules is known as universal grammar
  - an innate set of linguistic principles shared by all humans
  - attempts to explain language acquisition in general, not describe specific languages
  - proposes a set of rules intended to explain language acquisition in child development

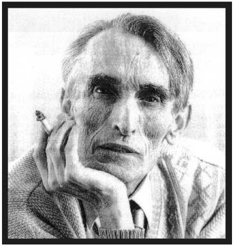
Marcel-Paul Schützenberger

- October 24, 1920 – July 29, 1996.
- French mathematician and Doctor of Medicine.
- Professor of the Faculty of Sciences, University of Paris
- Full Member of French Academy of Sciences.



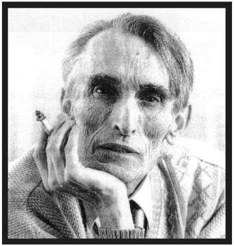
Biography

- First trained as a physician, doctorate in medicine in 1948.
- PhD in mathematics in 1953
- Professor at the University of Poitiers, 1957-1963
- Director of research at the CNRS, 1963-1964
- Professor in the Faculty of Sciences at the University of Paris, 1964-1996



Contribution

- Formal
  languages with Noam Chomsky
  - Chomsky-Schützenberger hierarchy
  - Chomsky-Schützenberger theorem
- Automata
  with Samuel Ellenberger
- Biology and Darwinism
  - Mathematical critique of neodarwinism (1966)



Alphabet

- In computational practice, data are encoded in the computer's memory as strings of bits or other symbols appropriate for manipulation by a computer.
- How to understand the mathematics of strings of symbols?
- **Definition**: An *alphabet* is a finite, nonempty set of symbols. We use Σ to denote this alphabet.
  - the Roman alphabet $\{a, b, \ldots, z\}$.
  - binary alphabet $\{0, 1\}$:
    the alphabet particularly pertinent to the theory of computation.
- Note: any object can be in an alphabet.
- In a formal point of view, an alphabet is simply a finite set of any sort.
- For simplicity, however, we use as symbols only letters, numerals, and other common characters such as \$, or $\#$.

String

- A *string* is a finite sequence of symbols from Σ.
- We simply juxtapose the symbols to denote a string.
  - *watermelon* is a string over the alphabet $\{a, b, \ldots, z\}$.
  - 0111011 is a string over $\{0, 1\}$.
- A string has no symbols at all
  $\Rightarrow$ the **empty string**
  $\Rightarrow$ denoted by ϵ.
- We generally use u, v, x, y, z, and Greek letters to denote strings.
  - Example: we use *w* as a name for the string *abc*.

String

- The **length** of a string s, denoted $|s|$, is the number of symbols in it.
  - $|101| = 3$
  - $|\epsilon| = 0$
- A string $w \in \Sigma^*$, can be considered as a function $w : \{1, \ldots, |w|\} \rightarrow \Sigma$,
  the value of $w(j)$, where $1 \leq j \leq |w|$, is the symbol in the jth position of w.
  - Example: if w = *accordion*, then $w(3) = w(2) = c$, and $w(1) = a$.
- **Occurrences** of a symbol in a string:
  the symbol $\sigma \in \Sigma$ occurs in the jth position of the string $w \in \Sigma^*$ if $w(j) = \sigma$.

Operations on strings

- **Concatenation**: $x \circ y$ (or simply xy)
  - Formally, $w = x \circ y$ iff $|w| = |x| + |y|$, $w(j) = x(j)$ for $j = 1, \ldots, |x|$, and $w(|x| + j) = y(j)$ for $j = 1, \ldots, |y|$.
  - Example: $01 \circ 001 = 01001$, $beach \circ boy = beathboy$
  - Note: $w \circ \epsilon = \epsilon \circ w$ for any string $|w|$
  - Associative: $(wx)y = w(xy)$ for any strings w, x and y.
- A string v is a **substring** of a string w iff there are strings x and y such that $w = xvy$.
  - Note: every string is a substring of itself. ($x = \epsilon$ and $y = \epsilon$)
  - ϵ is a substring of every string.
- If $w = xv$ for some x, then v is a **suffix** of w.
- If $w = vy$ for some y, then v is a **prefix** of w.
- Example: *road* is a prefix of *roadrunner*, a suffix of *abroad*, and a substring of both these and of *broader*.

Operations on strings

- **Definition** w^i:
  for each string w and each natural number i, the string w^i is defined as

$$w^0 = \epsilon, \text{ the empty string}$$
$$w^{i+1} = w^i \circ w, \text{ for each } i \geq 0$$

- Definition by induction : w^{i+1} is defined in terms of w^i.
  - Example: $(do)^2 = ?$
  - $(do)^2 = (do)^1 \circ do$, $(i = 1)$
  - $(do)^1 = (do)^0 \circ do$, $(i = 0)$
  - $(do)^0 = \epsilon$
  - $\Rightarrow (do)^2 = (e \circ do) \circ do = dodo$

Operations on strings

- **Definition** w^R: **reversal** of a string w, denoted by w^R, is defined as
  - If $|w| = 0$, then $w^R = w = \epsilon$
  - If $|w| = n + 1 \geq 0$, then $w = ua$ for some $a \in \Sigma$, and $w^R = au^R$
- The string "spelled backwards":
  - Example: $reverse^R = esrever$.
- Note: $(wx)^R = x^R w^R$, for any strings w and x.

Languages definition

- Σ^* denotes the set of all sequences of strings that are composed of zero or more symbols of Σ.
- Σ^+ denotes the set of all sequences of strings composed of one or more symbols of Σ.
  - $\Sigma^+ = \Sigma^* - \{\epsilon\}$
- A *language* is a subset of Σ^*.
- A language is a collection of sentences of finite length all constructed from a finite alphabet of symbols.
  - Σ^*, $\emptyset$ and Σ are languages.
  - $\{aba, czr, d, f\}$ is a language over $\{a, b, \ldots, z\}$.
- The infinite languages are specified by the scheme:

$$L = \{w \in \Sigma^* : \text{ w has property } P\}$$

  - $\{w \in \{0,1\}^* : w \text{ has an equal number of 0's and l's}\}$
  - $\{w \in \Sigma^* : w = w^R\}$

Operations on languages

- **Concatenation**: if L_1 and L_2 are languages over Σ, their concatenation is $L = L_1 \circ L_2$, or simply $L = L_1 L_2$, where
$$L = \{w \in \Sigma^* : w = x \circ y \text{ for some } x \in L_1 \text{ and } y \in L_2\}$$
  - Example: $\Sigma = \{0, 1\}$,
    $L_1 = \{w \in \Sigma^* : w \text{ has an even number of 0's}\}$,
    $L_2 = \{w \in \Sigma^* : w \text{ starts with a 0 and the rest of the symbols are l's}\}$
  - $\Rightarrow L_1 \circ L_2 = \{w \in \Sigma^* : w \text{ has an odd number of 0's}\}$

Operations on languages

- **Kleene star** of a language L, denoted by L^*, is the set of all strings obtained by concatenating zero or more strings from L:

$$L^* = \{w \in \Sigma^* : w = w_1 \circ \cdots \circ w_k \text{ for some } k \geq 0 \text{ and some } w_1, \ldots, w_k \in L\}$$

  - Example: $\Sigma = \{0, 1\}$, $L = \{01, 1, 100\}$, $\Rightarrow 110001110011 \in L^*$
  - since $110001110011 = 1 \circ 100 \circ 01 \circ 1 \circ 100 \circ 1 \circ 1$, and each of these strings is in Ls.

Grammars

- A **grammar** can be regarded as a device that enumerates the sentences of a language.
- Types of grammars
  - **Prescriptive**: prescribes authoritative norms for a language
  - **Descriptive**: attempts to describe actual usage rather than enforce arbitrary rules
  - **Formal**: a precisely defined grammar, such as context-free
  - **Generative**: a formal grammar that can generate natural language expressions

Formal grammars

- Two broad categories of formal languages: **generative** and **analytic**
- A **generative** grammar formalizes an algorithm that generates valid strings in a language
- An **analytic** grammar is a set of rules to reduce an input string to a boolean result that indicates the validity of the string in the given language
- A generative grammar describes how to **write** a language, and an analytic grammar describes how to **read** it (a parser).

Formal Grammars

- How to determine whether a string is valid within a given langauge?
- A **grammar** is a set of formation rules that describe which strings formed from the alphabet of a language are syntactically valid within the language.
  - Grammars are composed of:
  - a set of string rewriting rules
  - an assigned start symbol
- The language described by a grammar is the set of strings that can be generated by applying these **rules** arbitrarily, starting with the **start symbol**.
- Grammars could be thought of *language generators* or *string recognizers for languages*.

Example

- Assume the alphabet consists of a and b, the start symbol is S, rules are:
  1. $S \longrightarrow aSb$
  2. $S \longrightarrow ba$
- Operations: start with S, and choose a rule to apply to it
  - choose rule 1, obtain the string aSb
  - choose rule 1 again, replace S with $aSb \Rightarrow aaSbb$
  - repeated at will until all occurrences of S are removed, and only symbols from the alphabet remain (i.e., a and b).
  - choose rule 2, replace S with $ba \Rightarrow aababb$
  - $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aababb$
- The language of the grammar is the set of all the strings that can be generated using this process:
  - $\{ba, abab, aababb, aaababbb, \ldots\}$

Grammars

- Defined as a quadruple $\langle V_t, V_n, P, S \rangle$, where:
- V_t = the *terminal vocabulary*
  - i.e. the words of the language: finite (but productive ...)
- V_n = the *non-terminals*
  - i.e. the set of categories, used for the benefit of the grammar to express generalizations over items in V_t
- P = the *set of productions*,
  - i.e. rules in the grammar: finite, and of the form

    $$\alpha \rightarrow \beta$$

    where $\alpha, \beta \in (V_n \cup V_t)$
- S = the *distinguished symbol*, or 'start' symbol ('sentence')
  - $S \in V_n$,
  - P must include at least one rule $\alpha \rightarrow \beta$ where $\alpha = S$.

Generative

- A grammar is *generative* if it predicts (explicitly defines, or characterizes) *all and only all* strings $\in$ the language.
- The language of a grammar G, denoted as $L(G)$, is defined as all those strings over Σ that can be generated by G.
  - $L(G) = \{w \in \Sigma^* : S \Rightarrow_G^* w\}$

Example

- Consider the grammar G where $V_n = \{S, B\}$, $V_t = a, b, c$, S is the start symbol, and P consists of :
  1. $S \longrightarrow aBSc$
  2. $S \longrightarrow abc$
  3. $Ba \longrightarrow aB$
  4. $Bb \longrightarrow bb$
- Some examples of the derivation of strings in $L(G)$ are:
  - $S \Rightarrow_2 abc$
  - $S \Rightarrow_1 aBSc \Rightarrow_2 aBabcc \Rightarrow_3 aaBbcc \Rightarrow_4 aabbcc$
  - $S \Rightarrow_1 aBSc \Rightarrow_1 aBaBScc \Rightarrow_2 aBaBabccc \Rightarrow_3 aaBBabccc \Rightarrow_3 aaBaBbccc \Rightarrow_3 aaaBBbccc \Rightarrow_4 aaaBbbccc \Rightarrow_4 aaabbbccc$

Chomsky Hierarchy

Type	Grammar	Language	Automata
3	Finite State	Regular	Finite
2	Context-Free	C-F	Pushdown
1	Context-Sensitive	C-S	Linear-Bounded
0	General Rewrite	Unrestricted	Turing Machines

- occasionally referred to as Chomsky–Schützenberger hierarchy
- described by Noam Chomsky in 1956
- hierarchy of grammars
  - Type 3 $\subset$ Type 2 $\subset$ Type 1 $\subset$ Type 0

Type-3 grammars

Type	Grammar	Language	Automata
3	Finite State	Regular	Finite
2	Context-Free	C-F	Pushdown
1	Context-Sensitive	C-S	Linear-Bounded
0	General Rewrite	Unrestricted	Turing Machines

- *regular grammars*: $A \longrightarrow a$, and $A \longrightarrow aB$
  - a single nonterminal on the left-hand side.
  - a single terminal, possibly followed (or preceded, but not both) by a single nonterminal on the right-hand side.
  - $S \longrightarrow \epsilon$ is allowed if S is NOT on the right side of any rule
- generate the *regular languages*:
  - can be decided by *finite state automata*.
  - can be obtained by *regular expressions* (search patterns & lexical structure of programming languages)

Type-2 grammars

Type	Grammar	Language	Automata
3	Finite State	Regular	Finite
2	Context-Free	C-F	Pushdown
1	Context-Sensitive	C-S	Linear-Bounded
0	General Rewrite	Unrestricted	Turing Machines

- *context-free grammars*: $A \rightarrow \gamma$
  - a single nonterminal on the left-hand side.
  - a string of terminals and nonterminals on the right-hand side. (less restrictive than Type-3)
- generate the *context-free languages*:
  - can be recognized by *non-deterministic pushdown automata*.
  - theoretical basis for the syntax of most programming languages.

Type-1 grammars

Type	Grammar	Language	Automata
3	Finite State	Regular	Finite
2	Context-Free	C-F	Pushdown
1	Context-Sensitive	C-S	Linear-Bounded
0	General Rewrite	Unrestricted	Turing Machines

- *context-sensitive grammars*: $\alpha A \beta \rightarrow \alpha \gamma \beta$
  - α and β may be empty.
  - A is a single nonterminal. (less restrictive than Type-2)
  - γ is a nonempty string of nonterminals and terminals. (less restrictive)
  - $S \rightarrow \epsilon$ is allowed if S is NOT on the right side of any rule.
- generate the *context-sensitive languages*:
  - can be recognized by *linear bounded automata* (a nondeterministic Turing machine whose tape is bounded by a constant times the length of the input).

Type-0 grammars

Type	Grammar	Language	Automata
3	Finite State	Regular	Finite
2	Context-Free	C-F	Pushdown
1	Context-Sensitive	C-S	Linear-Bounded
0	General Rewrite	Unrestricted	Turing Machines

- unrestricted grammars: $\alpha \rightarrow \beta$
- generate the *recursively enumerable languages*:
  - can be recognized by *Turing machines*.

Exercise 1

- Given $\Sigma = \{0, 1\}$, please provide the specifications for the following languages:
- The language consists of all strings begin with 0.
  - $\{0\}\{0, 1\}^*$
- The language consists of all strings begin with 0, and end with 1.
  - $\{0\}\{0, 1\}^*\{1\}$
- The language consists of all strings with odd lengths.
  - $\{0, 1\}^{2n-1}, n = 1, 2, \ldots$
- The language consists of all strings with substring of three consecutive 0.
  - $\{0, 1\}^*000\{0, 1\}^*$
- The language consists of all strings without substring of three consecutive 0.
  - $\{001, 01, 1\}^*$

Exercise 2

- Consider the grammar G where $V_n = \{S, B, C\}$, $V_t = a, b, c$, S is the start symbol, and P consists of :
  1. $S \longrightarrow aBC$
  2. $S \longrightarrow aSBC$
  3. $aB \longrightarrow ab$
  4. $bB \longrightarrow bb$
  5. $CB \longrightarrow BC$
  6. $bC \longrightarrow bc$
  7. $cC \longrightarrow cc$

  try to provide one derivation for sentence $aaabbbccc$.
  - $S \Rightarrow_2 aSBC \Rightarrow_2 aaSBCBC \Rightarrow_1 aaaBCBCBC \Rightarrow_3 aaabCBCBC \Rightarrow_5 aaabBCCBC \Rightarrow_4 aaabbCCBC \Rightarrow_5 aaabbCBCC \Rightarrow_5 aaabbBCCC \Rightarrow_4 aaabbbCCC \Rightarrow_6 aaabbbcCC \Rightarrow_7 aaabbbccC \Rightarrow_7 aaabbbccc$

1. Personalities
   - Noam Chomsky
   - Marcel Schützenberger
2. Languages
   - Alphabets
   - Languages
3. Grammars
   - Introduction
   - Formal definition
   - Examples
4. Chomsky Hierarchy