Sets

A **set** is a collection or group of objects or *elements* or *members*. (Cantor 1895)

- A set is said to *contain* its elements.
- There must be an underlying universal set U, either specifically stated or understood.

Notation:

- list the elements between braces:

  $S = \{a, b, c, d\} = \{b, c, a, d, d\}$
- specification by predicates:

  $S = \{x | P(x)\}$,

  S contains all the elements from U which make the predicate P true.
- brace notation with ellipses:

  $S = \{..., -3, -2, -1\}$,

  the negative integers.

Common Universal Sets

- $\mathbb{R}$ = reals
- $\mathbb{N}$ = natural numbers = $\{0, 1, 2, 3, ...\}$, the *counting* numbers
- $\mathbb{Z}$ = integers = $\{.., -3, -2, -1, 0, 1, 2, 3, 4, ..\}$
- $\mathbb{Q}$ = rationals = m/n, where $m, n \in \mathbb{Z}, n \neq 0$

Note already that some sets are *infinite* ...

Notation:

x is a member of S or x is an element of S:

$$x \in S$$

x is not an element of S:

$$x \notin S$$

Subsets

Definition: The set A is a *subset* of the set B, denoted $A \subseteq B$, iff

$$\forall x : x \in A \Rightarrow x \in B$$

Definition: The *void* set, the *null* set, the *empty* set, denoted $\emptyset$, is the set with no members.

Note:

- the assertion $x \in \emptyset$ is always false.
- $\emptyset$ is a subset of every set.
- A set B is always a subset of itself.

Definition: If $A \subseteq B$ but $A \neq B$ then we say A is a *proper* subset of B, denoted $A \subset B$ (in some texts).

Definition: The set of all subsets of a set A, denoted $\wp(A)$, is called the *power set* of A.

Example: If $A = \{a, b\}$ then:

$$\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Cardinality

Definition: The number of (distinct) elements in A, denoted $|A|$, is called the *cardinality* of A.

If the cardinality is a natural number (in $\mathbb{N}$), then the set is called *finite*, otherwise it is *infinite*.

Example:

$$A = \{a, b\}$$

$$|\{a, b\}| = 2$$

$$|\wp(\{a, b\})| = 4$$

A is finite and so is $\wp(A)$.

Useful Fact: $|A| = n$ implies $|\wp(A)| = 2^n$

$\mathbb{N}$ is infinite since $|\mathbb{N}|$ is not a natural number. It is called a *transfinite cardinal number*.

Operations on Sets

Union: $A \cup B \;=\; \{x : x \in A \;\vee\; x \in B\}$

$A = \{a, b, c\},\ B = \{c, d, e\},\ A \cup B = \{a, b, c, d, e\}$

Intersection: $A \cap B \;=\; \{x : x \in A \;\wedge\; x \in B\}$

$A = \{a, b, c\},\ B = \{c, d, e\},\ A \cap B = \{c\}$

Difference: $A - B \;=\; \{x : x \in A \;\wedge\; x \notin B\}$

$A = \{a, b, c\},\ B = \{c, d, e\},\ A - B = \{a, b\}$

Symmetric Difference: $A \Delta B = (A - B) \cup (B - A)$

$A = \{a, b, c\},\ B = \{c, d, e\},\ A \Delta B = \{a, b, d, e\}$

Sets are *disjoint* if $A \cap B = \emptyset$.

Power Set of A (2^A) is set of all possible subsets of A, including $\emptyset$ and A itself. If $A = \{a, b, c\}$, then $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

Pairs, n-Tuples and Relations

$\langle x, y\rangle$ is the *ordered pair* of *x* and *y*.

Note: $\langle x, y\rangle \neq \{x, y\}$ as the latter is unordered.

Definition: The Cartesian product of A with B, denoted $A \times B$, is the set of *ordered pairs* $\{<a, b> | a \in A \wedge b \in B\}$

Example:

$A = \{a, b\}$

$B = \{1, 2, 3\}$

$A \times B = \{<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>\}$

What is $B \times A$?

$A \times B \times A$?

If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

Pairs, n-Tuples and Relations

Any subset R of $A \times B$ is a *binary relation* between A and B. Given a binary relation R, we can associate with it a predicate $R(x, y)$ which is true iff $\langle x, y \rangle \in R$.

Extending pairs to *n*-tuples (triples, etc.) is straightforward.

$\langle x_1 \dots x_n \rangle$ is a sequence of ordered *n*-tuples.

The Cartesian product of *n* sets $A_1 \dots A_n$ is defined as $A_1 \times A_2 \times \dots \times A_n = \{\langle a_1, a_2 \dots a_n \rangle : (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \dots \wedge (a_n \in A_n)\rangle$.

Any subset R of $A_1 \times A_2 \times \dots \times A_n$ is an *n-ary relation* between the *n* sets $A_1 \dots A_n$. Given an *n*-ary relation R, we can associate with it a predicate $R(x_1 \dots x_n)$ which is true iff $\langle x_1 \dots x_n \rangle \in R$.

Functions

A *function* (*mapping*, *map*) f from set A to set B is denoted by $f : A \to B$

f associates with each x in A one and only one y in B.

A is called the *domain* and B is called the *codomain*.

The *range* of f, denoted by $f(A)$, is the set of all images of points in A under f.

Given a binary relation $R \subseteq A \times B$, the inverse relation of R is defined as $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$. Obviously $(R^{-1})^{-1} = R$.

Similarly, given a function $f : A \to B$, it admits an inverse function $f^{-1} : B \to A$ if the following identity holds true: $f(a) = b \leftrightarrow f^{-1}(b) = a$.

Note: a binary relation R always admits a unique inverse relation R^{-1} while only the injective functions admit an inverse function.

Injections, Surjections and Bijections

Let f be a function from A to B.

Definition: f is *one-to-one* (denoted 1-1) or *injective* if $a \neq b$ implies $f(a) \neq f(b)$

Definition: f is *onto* or *surjective* if $f(A) = B$

Definition: f is *bijective* if it is surjective and injective (one-to-one and onto).

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$

Definition: Let $f : B \rightarrow C, g : A \rightarrow B$. The *composition* of f with g, denoted $f \circ g$, is the function from A to C defined by $f \circ g(x) = f(g(x))$

Countability

Definition: If a set has the same cardinality as a subset of the natural numbers $\mathbb{N}$, then the set is called *countable*.

If $|A| = |\mathbb{N}|$, the set A is *countably infinite* or *denumerable*. The (transfinite) cardinal number of the set $\mathbb{N}$ is *aleph null* $= \aleph_0$.

If a set is not countable we say it is *uncountable*.
The following sets are uncountable (we show later):

- The real numbers in [0, 1]
- $\wp(\mathbb{N})$, the power set of $\mathbb{N}$
- The set of functions from $\mathbb{N}$ to $\mathbb{N}$

Note: With infinite sets proper subsets can have the same cardinality. This cannot happen with finite sets.

Countability carries with it the implication that there is a *listing* or *enumeration* of the elements of the set.

Examples

Theorem: $|A| \leq |B|$ if there is an injection from A to B.

Example: if A is a subset of B then $|A| \leq |B|$

Proof: the function $f(x) = x$ is an injection from A to B

Theorem: $|A| = |B|$ iff there is is a bijection from A to B

Example: $|\mathbb{E}| = |\mathbb{N}|$, where $\mathbb{E}$ is the set of even integers (even though $\mathbb{E}$ is a proper subset of $|\mathbb{N}|$)

Proof: Let $f(x) = 2x$. Then f is a bijection from $\mathbb{N}$ to $\mathbb{E}$:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & \ldots \end{array}$$