CA200 – Quantitative Analysis for Business Decisions

File name: CA200_Section_02A_Probability

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2. **Probability & Decision making**

2.1 What do we mean by probability?

We will often use the term **event** to mean some occurrence such as

“drawing an ace from a deck of cards”

“getting black in a roulette roll”

We define the probability of an event E as

$$p(E) = \frac{\textit{Number of favourable outcomes}}{\textit{Total number of possible outcomes}}$$

Example 1: Let E be the event of “drawing a heart from an ordinary deck of cards”. Then,

$$p(E) = \frac{13}{52} = \frac{1}{4}$$

We can see that, in this definition, probabilities are regarded as **relative frequencies**.

Example 2: Let E be “the choosing of a number on the interval [0, 1] such that the number is less than 1/3.”

In this case, the use of the definition is a bit more subtle but essentially it gives

$$p(E) = \frac{\textit{Length of } [0, 1/3)}{\textit{Length of } [0, 1]} = \frac{1}{3}$$

Note: When the probability of an event is based on past data and the circumstances are repeatable by test then this probability is known as **statistical probability** or **objective probability**. However, very often we have to instead use **subjective probability** such as “there is a 40% chance (i.e. p = 0.4) of winning a contract that our company has tendered for”.

2.2 Basic rules of probability; Some initial examples

Definitions:

(a) **Independent events** do not affect each other's occurrence.

E.G. last week's lottery numbers don't impact on this week's numbers.

(b) **Mutually exclusive events** are events that cannot happen at the same time.

E.G. a card drawn from a deck cannot be both a spade and a diamond.

Rules of probability:

(a) **Multiplication rule for independent events** X and Y:

p(X and Y) = p(X) x p(Y)

e.g.

(1) p(throwing 3 followed by 6 in ***2*** rolls of a die) =

p(throwing 3) x p(throwing 6) = $\frac{1}{6} x \frac{1}{6} = \frac{1}{36}$

However,

(2) p(throwing 3 and 6 in ***2*** rolls of a die, regardless of order) =

p(throwing 3 followed by 6) + p(throwing 6 followed by 3) [Use of Rule (b)]

$= \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$

(b) **Addition rule for mutually exclusive events** X and Y:

p(X or Y) = p(X) + p(Y)

e.g.

p(throwing a 3 or a 6 in roll of a die) = p(throwing 3) + p(throwing 6) = 1/6+1/6 = 1/3

(b*) **Addition rule for any two events** X and Y:

p(X or Y or both) = p(X) + p(Y) – p(X and Y)

e.g. p("drawing a Heart or a King from an ordinary deck of cards") =

p("drawing a Heart") + p("drawing a King") - p("drawing the King of Hearts")

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$ [*The subtraction is to avoid double counting*]

Note: Rule (b*) becomes Rule (b) for mutually exclusive events where p(X & Y) = 0

(c) **Conditional prob. p(X|Y) = "Prob. of X given that Y has occurred"**:

$$p(X \mid Y) = \frac{p(X \text{ and } Y)}{p(Y)}$$

Note: It follows from (c) that p(X and Y) = p(Y)p(X|Y) = p(X)p(Y|X) (by symmetry)

So, we have the important formulae [***see following section – Bayes' Rule***]

$$p(X \mid Y) = \frac{p(X)p(Y \mid X)}{p(Y)} \text{ and } p(Y \mid X) = \frac{p(Y)p(X \mid Y)}{p(X)}$$

Example (from Section: 1.4.1)

From past experience it is known that a machine is set up correctly on 90% of occasions. If the machine is set up correctly then the conditional probability of a good part is 95% but if the machine is not set up correctly then the conditional probability of a good part is only 30%.

On a particular day the machine is set up and the first component produced and found to be good. ***What is the probability that the machine is set up correctly?*** [*The answer turns out to be 0.966*]

Solution

In this example let X = event the "machine is set up correctly

Y = event that "first component is good"

So we need to find $p(X \mid Y)$ using Rule (c).

We are given the three pieces of information,

$p(X) = 0.9$ (and hence that its "complement" $p(\overline{X}) = 0.1$)

$$p(Y \mid X) = \frac{p(X \text{ and } Y)}{p(X)} = 0.95$$

$$p(Y \mid \overline{X}) = \frac{p(\overline{X} \text{ and } Y)}{p(\overline{X})} = 0.3$$

Now, by Rule (b), $p(Y) = p(X \text{ and } Y) + p(\overline{X} \text{ and } Y)$

=> $p(Y) = 0.95 * p(X) + 0.3 * p(\overline{X})$

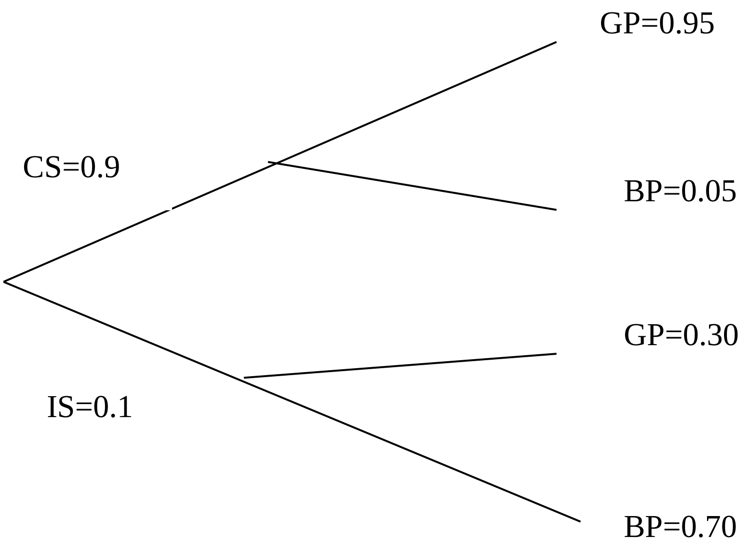
=> $p(Y) = 0.95 * 0.9 + 0.3 * 0.1 = 0.885$

Hence, by Rule (c),

$$p(X \mid Y) = \frac{0.95 * 0.9}{0.885} = 0.966$$ [***The answer!***]

Note: Probability trees can be helpful in working out problems like the one above.

Let CS = Correct Setup, IS = Incorrect setup, GP = Good Part and BP = Bad Part. then, we can draw the following diagram,



Then,

CSGP	= 0.9*0.95 = 0.855
CSBP	= 0.9*0.05 = 0.045
ISGP	= 0.1*0.3 = 0.03
ISBP	= 0.1*0.7 = 0.07
Probability total	= 1.0

2.3 More applications of probability

Example 1: *Note use of table of probabilities as an aid*.

There are 100 students in a first year class, of which 36 are male and studying accounting, 9 are male and not studying accounting, 42 are female and studying accounting, and 13 are female and not studying accounting. Use these data to deduce probabilities concerning a student drawn at random.

Solution: We can draw the table

	Accounting A	Not Accounting $\bar{A}$	**Total**
Male M	36	9	
Female F	42	13	
Total			

Evidently, we can calculate the row and column totals immediately (note that A and $\bar{A}$ are mutually exclusive, as are M and F). This gives

	Accounting A	Not Accounting $\bar{A}$	**Total**
Male M	36	9	45
Female F	42	13	55
Total	78	22	100

A. No restriction on student picked at random

(A1)From the table, it is easily seen (by our definition of probability) that

p(M)	p(F)	p(A)	p($\bar{A}$)
0.45	0.55	0.78	0.22

(A2) We can also obtain the "joint" probabilities:

p(M and A)	p(M and $\bar{A}$)	p(F and A)	p(F and $\bar{A}$)
0.36	0.09	0.42	0.13

(A3) From (A1) and (A2) it is easily seen that

p(M) = p(M&A)+p(M& $\bar{A}$)	p(F) = p(F&A)+p(F& $\bar{A}$)	p(A) = p(M&A)+p(F&A)	p($\bar{A}$) = p(M& $\bar{A}$)+p(F& $\bar{A}$)

B. Student picked is male => female section of table is removed [*Conditional*]

By Rule (c) p(A|M) = p(A and M)/p(M) = 0.36/0.45 = 0.80.

C. Student picked does accountancy [*Conditional*]

By Rule (c) p(M|A) = p(A and M)/p(A) = 0.36/0.78 = 0.462.

Note: we can see that, in general, **p(M|A) is not equal to p(A|M)**.

<u>Example 2</u>: *Note use of table of probabilities as an aid*.

A company has three production sections S1, S2 and S3 that contribute 40%, 35% and 25%, respectively, to total output. The following percentages of faulty units have been observed:

S1	2% (0.02)
S2	3% (0.03)
S3	4% (0.04)

There is a final check before output is dispatched. Calculate the probability that a unit found faulty at this check has come from S1.

<u>Solution</u>:

Let F denote the event that a unit has been found faulty (in the final check).

Let p(S1), p(S2) and p(S3) be the probability that a unit chosen at random comes from S1, S2 and S3, respectively.

We are given that p(S1) = 0.40, p(S2) = 0.35 and p(S3) = 0.25.

We are also given that p(F|S1) = 0.02, p(F|S2) = 0.03 and p(F|S3) = 0.04.

<u>What is required to be found is p(S1|F)</u>.

From Rule (c) we have $p(S1 \mid F) = \dfrac{p(S1\ and\ F)}{p(F)} = \dfrac{p(S1)\,p(F \mid S1)}{p(F)}$

In fact, we also have $p(S2 \mid F) = \dfrac{p(S2\ and\ F)}{p(F)} = \dfrac{p(S2)\,p(F \mid S2)}{p(F)}$ and

$$p(S3 \mid F) = \frac{p(S3\ and\ F)}{p(F)} = \frac{p(S3)\,p(F \mid S3)}{p(F)}$$

As a faulty part can only come from S1, S2 or S3 (mut. exclusive) we must have that

p(S1|F) +p(S2|F) + p(S3|F) = 1.0

Therefore, by addition,

$$\frac{p(S1)\,p(F \mid S1)}{p(F)} + \frac{p(S2)\,p(F \mid S2)}{p(F)} + \frac{p(S3)\,p(F \mid S3)}{p(F)} = 1.0$$

Hence, multiplying across by p(F) gives

p(S1)p(F|S1) + p(S2)p(F|S2) + p(S3)p(f|S3) = p(F)

Therefore, 0.40*0.02+0.35*0.03+0.25*0.04=0.0080+0.0105+0.0100=0.0285 = p(F)

Hence, the required answer is

$$p(S1 \mid F) = \frac{0.40 * 0.02}{0.0285} = 0.2807$$ (*So, 28.07% chance that faulty unit is from S3*)

Exercise: Show that p(S2|F) = 0.3684 and p(S3|F) = 0.3509.

Note: It is clear that these values satisfy p(S1|F) +p(S2|F) + p(S3|F) = 1.0

2.4 Bayes' Rule (or Theorem)

The formula

$$p(X \mid Y) = \frac{p(X)p(Y \mid X)}{p(Y)}$$

is known as Bayes' Rule. As the last example shows this allows one to work backwards from effect ("faulty unit found") to cause ("came from S1").

We will make use of this formula in section 3, when using decision trees. There information is given in the form of conditional probabilities and the reverse of these probabilities must be found.

2.5 Permutations (see CA200_Section_02B_Probability)

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2.6 Combinations (see CA200_Section_02B_Probability)

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2.7 Expected value (see CA200_Section_02B_Probability)

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2.8 Value of Perfect Information (see CA200..02B_Probability)

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2.9 Alternative decision rules (see CA200..02B_Probability)

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2.10 Summary (see CA200..02B_Probability)

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2.11 Exercises (see CA200..02B_Probability)

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