Theorems of E3

J.M. Morris & A. Bunkenburg

University of Glasgow

Notation

E3 supports undefined. E, F, G, ..., and t to stand for terms of arbitrary type. T, U, ... stand for arbitrary types (including the booleans). P, Q, and R stand for formulae (i.e. terms of type $\mathbb{B}$).

Equivalence and nonequivalence

$\equiv$-refl:	$E \equiv E$	
$\equiv$-symm:	$(E \equiv F) \equiv (F \equiv E)$	
$\equiv$-trans:	$(E \equiv F) \wedge (F \equiv G) \Rightarrow (E \equiv G)$	
$\equiv$-truth:	$((E \equiv F) \equiv \text{True}) \equiv (E \equiv F)$	
$\equiv$-non-truth:	$((E \equiv F) \not\equiv \text{True}) \equiv (E \not\equiv F)$	
$\equiv$-Falsity:	$((E \equiv F) \equiv \text{False}) \equiv (E \not\equiv F)$	
$\not\equiv$-defn:	$(E \not\equiv F) \equiv \neg(E \equiv F)$	
$\not\equiv$-symm:	$(E \not\equiv F) \equiv (F \not\equiv E)$	
strong $\equiv$:	$(E\equiv F) \vee (E\not\equiv F)$	
$\equiv$-truth:	$\Delta(E\equiv F)$ and $\Delta(E\not\equiv F)$	

Boolean equivalence and nonequivalence

establishment:	$(P \equiv \text{True}) \equiv \Delta P \wedge P$	
$\equiv$-unit:	$(P \equiv \text{True}) \equiv P$	if ΔP
establishment:	$(P \equiv \text{False}) \equiv \Delta P \wedge \neg P$	
$\equiv$-$\neg$:	$(P \equiv \text{False}) \equiv \neg P$	if ΔP
$\Delta\mathbb{B}$:	$((P \equiv \text{True}) \equiv P) \equiv \Delta P$	
$\equiv$-assoc:	$((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$	if ΔP, ΔQ, ΔR
$\not\equiv$-assoc:	$((P \not\equiv Q) \not\equiv R) \equiv (P \not\equiv (Q \not\equiv R))$	if ΔP, ΔQ, ΔR
$\equiv$-$\not\equiv$-assoc	$((P \not\equiv Q) \equiv R) \equiv (P \not\equiv (Q \equiv R))$	if ΔP, ΔQ, ΔR
$\equiv$-$\not\equiv$-assoc	$((P \equiv Q) \not\equiv R) \equiv (P \equiv (Q \not\equiv R))$	if ΔP, ΔQ, ΔR
$\equiv$-$\mathbb{B}$:	$(P\equiv Q) \equiv (P\vee\neg Q) \wedge (\neg P\vee Q)$	if ΔP, ΔQ
$\equiv$-$\mathbb{B}$:	$(P\equiv Q) \equiv (P\wedge Q) \vee (\neg P\wedge\neg Q)$	if ΔP, ΔQ
truth cases:	$(P\equiv Q) \equiv ((P\equiv\text{True}) \equiv (Q\equiv\text{True})) \wedge ((P\equiv\text{False}) \equiv (Q\equiv\text{False}))$	

Substitution

Leibniz:	$(E \equiv F) \Rightarrow (G[x:=E] \equiv G[x:=F])$	
$\wedge$-subst:	$(E \equiv F) \wedge P[x:=E] \ \equiv\ (E \equiv F) \wedge P[x:=F]$	
$\vee$-subst:	$(E \not\equiv F) \vee P[x:=E] \ \equiv\ (E \not\equiv F) \vee P[x:=F]$	
$\Rightarrow$-subst:	$(E \equiv F) \Rightarrow P[x:=E] \ \equiv\ (E \equiv F) \Rightarrow P[x:=F]$	
$\wedge$-subst:	$(P \equiv \text{True}) \wedge Q[x:=E] \ \equiv\ (P \equiv \text{True}) \wedge Q[x:=F]$ if $P \Rightarrow (E \equiv F)$	
$\Rightarrow$-subst:	$P \Rightarrow Q[x:=E] \ \equiv\ P \Rightarrow Q[x:=F]$ if $P \Rightarrow (E \equiv F)$	

Shannon:	$Q[x{:=}P] \equiv (P \wedge Q[x{:=}True]) \vee (\neg P \wedge Q[x{:=}False])$	if ΔP

Negation

exchange:	$(\neg P \equiv Q) \equiv (\neg Q \equiv P)$	
False-defn:	$False \equiv \neg True$ and $\neg False \equiv True$	
$\neg$-exchange:	$(\neg P \equiv Q) \equiv (P \equiv \neg Q)$	
$\neg$-inv:	$\neg\neg P \equiv P$	
$\equiv$-mirror :	$(P \equiv Q) \equiv (\neg P \equiv \neg Q)$	
	$P \not\equiv \neg P$	if ΔP
$\neg\equiv$:	$\neg(P \equiv Q) \equiv (\neg P \equiv Q)$	if ΔP, ΔQ

Disjunction

$\vee$-symm:	$P \vee Q \equiv Q \vee P$	
$\vee$-assoc:	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	
$\vee$-idem:	$P \vee P \equiv P$	
$\vee$-zero:	$P \vee True \equiv True$ and $True \vee P \equiv True$	
$\vee$-unit:	$P \vee False \equiv P$ and $False \vee P \equiv P$	
$\vee/\vee$:	$P \vee (Q \vee R) \equiv (P \vee Q) \vee (P \vee R)$	
$\vee/\equiv$:	$(P \vee (Q \equiv R)) \equiv ((P \vee Q) \equiv (P \vee R))$	if ΔP
$\vee$-truth:	$(P \vee Q \equiv True) \equiv (P \equiv True) \vee (Q \equiv True)$	
$\vee$-falsity:	$(P \vee Q \equiv False) \equiv (P \equiv False) \wedge (Q \equiv False)$	
$\vee$-$\neg$truth:	$(P \vee Q \not\equiv True) \equiv (P \not\equiv True) \wedge (Q \not\equiv True)$	
$\vee$-$\neg$falsity:	$(P \vee Q \not\equiv False) \equiv (P \not\equiv False) \vee (Q \not\equiv False)$	
excluded false:	$(P \vee \neg P) \not\equiv False$	

Conjunction

$\wedge$-symm:	$(P \wedge Q) \equiv (Q \wedge P)$	
$\wedge$-assoc:	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	
$\wedge$-idem:	$P \wedge P \equiv P$	
$\wedge$-zero :	$P \wedge False \equiv False$ and $False \wedge P \equiv False$	
$\wedge$-unit:	$P \wedge True \equiv P$ and $True \wedge P \equiv P$	
$\wedge/\wedge$:	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$	
$\wedge/\not\equiv$:	$(P \wedge (Q \not\equiv R)) \equiv ((P \wedge Q) \not\equiv (P \wedge R))$	if ΔP
$\wedge$-truth:	$(P \wedge Q \equiv True) \equiv (P \equiv True) \wedge (Q \equiv True)$	
$\wedge$-falsity:	$(P \wedge Q \equiv False) \equiv (P \equiv False) \vee (Q \equiv False)$	
$\wedge$-$\neg$truth:	$(P \wedge Q \not\equiv True) \equiv (P \not\equiv True) \vee (Q \not\equiv True)$	
$\wedge$-$\neg$falsity:	$(P \wedge Q \not\equiv False) \equiv (P \not\equiv False) \wedge (Q \not\equiv False)$	
absorption:	$P \wedge (P \equiv Q) \equiv P \wedge Q$	if ΔP, ΔQ

Disjunction and conjunction

de Morgan	$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ and $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$	
$\vee/\wedge$:	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	
$\wedge/\vee$:	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
absorption:	$P \wedge (\neg P \vee Q) \equiv P \wedge Q$	if ΔP

absorption:	$P \lor (\neg P \land Q) \equiv P \lor Q$	if ΔP
absorption:	$P \land (P \lor Q) \equiv P$	
absorption:	$P \lor (P \land Q) \equiv P$	
consistency:	$(P \land Q \equiv P) \equiv (P \lor Q \equiv Q)$	

Boolean values

True	True	
negating False:	$\neg$False	
uniqeness:	False, True, and $\bot_{\mathbb{B}}$ are distinct from one another	
3-valued:	$(P \equiv \text{True}) \lor (P \equiv \text{False}) \lor (P \equiv \bot_{\mathbb{B}})$	

Implication

$\Rightarrow$-refl:	$P \Rightarrow P$	
True maximal:	$P \Rightarrow \text{True}$	
False minimal:	$\text{False} \Rightarrow P$	
$\Rightarrow$-trans:	$(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$	
transitivity:	$(P \equiv Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$	
transitivity:	$(P \Rightarrow Q) \land (Q \equiv R) \Rightarrow (P \Rightarrow R)$	
bi-implication:	$(P \Rightarrow Q) \land (Q \Rightarrow P) \equiv (P \equiv Q)$	if ΔP, ΔQ
$\Rightarrow$-connected:	$(P \Rightarrow Q) \lor (Q \Rightarrow P)$	
$\Rightarrow$-left-unit:	$\text{True} \Rightarrow P \equiv P$	
$\Rightarrow$-right-zero:	$P \Rightarrow \text{True} \equiv \text{True}$	
$\Rightarrow$-truth:	$(P \Rightarrow Q \equiv \text{True}) \;\equiv\; P \Rightarrow (Q \equiv \text{True})$	
$\Rightarrow$-truth:	$(P \Rightarrow Q \equiv \text{True}) \;\equiv\; (P \equiv \text{True}) \Rightarrow (Q \equiv \text{True})$	
contrapositive:	$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$	if ΔP, ΔQ
$\equiv$-weakening:	$(P \equiv Q) \Rightarrow (P \Rightarrow Q)$	
True-$\Rightarrow$:	$(P \equiv \text{True}) \Rightarrow Q \equiv P \Rightarrow Q$	

Implication and disjunction

$\Rightarrow$-defn:	$P \Rightarrow Q \equiv (P \not\equiv \text{True}) \lor Q$	
$\Rightarrow$-defn:	$P \Rightarrow Q \equiv \neg\Delta P \lor \neg P \lor Q$	
$\Rightarrow$-$\lor$:	$P \Rightarrow Q \equiv (P \lor Q \equiv Q)$	if ΔP, ΔQ
weakening:	$P \Rightarrow P \lor Q$	

Implication and conjunction

$\Rightarrow$-$\land$:	$P \Rightarrow Q \equiv (P \land Q \equiv P)$	if ΔP, ΔQ
weakening:	$P \land Q \Rightarrow P$	
shunting:	$P \land Q \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$	
modus ponens:	$P \land (P \Rightarrow Q) \Rightarrow Q$	
absorption:	$P \land (P \Rightarrow Q) \equiv (P \land Q)$	if ΔP
$\Leftrightarrow$-defn:	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \land (Q \Rightarrow P)$	

Implication and distribution

$\Rightarrow/\lor$:	$P \Rightarrow Q\lor R \equiv (P \Rightarrow Q) \lor (P \Rightarrow R)$	
$\Rightarrow/\land$:	$P \Rightarrow Q\land R \equiv (P \Rightarrow Q) \land (P \Rightarrow R)$	
$\Rightarrow/\Rightarrow$:	$P \Rightarrow (Q \Rightarrow R) \equiv (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	
$\Rightarrow/\equiv$:	$P \Rightarrow (Q \equiv R) \equiv (P \Rightarrow Q \equiv P \Rightarrow R)$	
$\Rightarrow/\equiv/\land$:	$P \Rightarrow (Q \equiv R) \;\equiv\; ((P \equiv True) \land Q \equiv (P \equiv True) \land R)$	
$\Rightarrow/\equiv/\land$:	$P \Rightarrow (Q \equiv R) \equiv (P \land Q \equiv P \land R)$	if ΔP
$\lor$-lub:	$P \lor Q \Rightarrow R \equiv (P \Rightarrow R) \land (Q \Rightarrow R)$	
$\land\backslash\Rightarrow$:	$P \land Q \Rightarrow R \equiv (P \Rightarrow R) \lor (Q \Rightarrow R)$	

Implication and monotonicity

$\Rightarrow$-right-mono:	$(P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))$
$\Rightarrow$-left-anti-mono:	$(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$
$\Rightarrow$-combination:	$(P \Rightarrow Q) \land (R \Rightarrow S) \Rightarrow (P \land R \Rightarrow Q \land S)$
$\Rightarrow$-combination:	$(P \Rightarrow Q) \land (R \Rightarrow S) \Rightarrow (P \lor R \Rightarrow Q \lor S)$
$\land$-mono:	$(P \Rightarrow Q) \Rightarrow (P \land R \Rightarrow Q \land R)$
$\land$-mono:	$(P \Rightarrow Q) \Rightarrow (R \land P \Rightarrow R \land Q)$
$\lor$-mono:	$(P \Rightarrow Q) \Rightarrow (P \lor R \Rightarrow Q \lor R)$
$\lor$-mono:	$(P \Rightarrow Q) \Rightarrow (R \lor P \Rightarrow R \lor Q)$

Properness

constants proper:	$(\forall x{:}T \bullet \Delta x)$
Δ-$\exists$:	$\Delta E \equiv (\exists x{:}T \bullet x{\equiv}E)$ x fresh
$\neg\Delta\bot$:	$\neg\Delta\bot_T$
$\bot$-improper:	$(\forall x{:}T \bullet x \not\equiv \bot_T)$
variables proper:	$\Delta\mathbf{x}_T$

Boolean properness

boolean properness:	$\Delta P \equiv (P \equiv True) \lor (P \equiv False)$
	$\Delta P \equiv \neg(P \Rightarrow \neg P) \lor \neg(\neg P \Rightarrow P))$
	$\Delta P \equiv (P \not\equiv \neg P)$
excluded middle:	$\Delta P \Leftrightarrow P \lor \neg P$
included middle:	$P \lor \neg P \lor \neg\Delta P$
	$\Delta True$ and $\Delta False$
$\Delta\neg$:	$\Delta(\neg P) \equiv \Delta P$
$\Delta\Delta$:	$\Delta\Delta P$

Range manipulation

trading:	$(\forall x{:}T \vert R \bullet P) \;\equiv\; (\forall x{:}T \bullet R \Rightarrow P)$
universal range:	$(\forall x{:}T \vert True \bullet P) \equiv (\forall x{:}T \bullet P)$
empty range:	$(\forall x{:}T \vert False \bullet P)$
range split:	$(\forall x{:}T \vert R\lor S \bullet P) \;\equiv\; (\forall x{:}T \vert R \bullet P) \land (\forall x{:}T \vert S \bullet P)$
partial trading:	$(\forall x{:}T \vert R\land S \bullet P) \;\equiv\; (\forall x{:}T \vert R \bullet S \Rightarrow P)$

trading:	$(\exists x{:}T \mid R \bullet P) \equiv (\exists x{:}T \bullet (R \equiv True) \wedge P))$
universal range:	$(\exists x{:}T \mid True \bullet P) \equiv (\exists x{:}T \bullet P)$
empty range:	$\neg(\exists x{:}T \mid False \bullet P)$
range split:	$(\exists x{:}T \mid R \vee S \bullet P) \equiv (\exists x{:}T \mid R \bullet P) \vee (\exists x{:}T \mid S \bullet P)$
partial trading:	$(\exists x{:}T \mid R \wedge S \bullet P) \equiv (\exists x{:}T \mid R \bullet (S \equiv True) \wedge P)$

Distribution of quantifications

$\forall/\wedge$:	$(\forall x{:}T \mid R \bullet P \wedge Q) \equiv (\forall x{:}T \mid R \bullet P) \wedge (\forall x{:}T \mid R \bullet Q)$
$\wedge/\forall$:	$(\forall x{:}T \mid R \bullet P \wedge Q) \equiv P \wedge (\forall x{:}T \mid R \bullet Q)$ if $(\exists x{:}T \bullet R \equiv True)$ (x not in P)
$\forall/\vee$:	$(\forall x{:}T \mid R \bullet P) \vee (\forall x{:}T \mid R \bullet Q) \Rightarrow (\forall x{:}T \mid R \bullet P \vee Q)$
$\vee/\forall$:	$P \vee (\forall x{:}T \mid R \bullet Q) \equiv (\forall x{:}T \mid R \bullet P \vee Q)$ (x not in P).
$\Rightarrow/\forall$:	$P \Rightarrow (\forall x{:}T \mid R \bullet Q) \equiv (\forall x{:}T \mid R \bullet P \Rightarrow Q)$ (x not in P)
$\forall/\equiv$:	$(\forall x{:}T \bullet P \equiv Q) \Rightarrow ((\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet Q))$
de Morgan:	$\neg(\forall x{:}T \mid R \bullet P) \equiv (\exists x{:}T \mid R \bullet \neg P)$
$\Delta\forall$:	$(\forall x{:}T \bullet \Delta P) \Rightarrow \Delta(\forall x{:}T \bullet P)$
$\exists/\vee$:	$(\exists x{:}T \mid R \bullet P \vee Q) \equiv (\exists x{:}T \mid R \bullet P) \vee (\exists x{:}T \mid R \bullet Q)$
$\vee/\exists$:	$(\exists x{:}T \mid R \bullet P \vee Q) \equiv P \vee (\exists x{:}T \mid R \bullet Q)$ if $(\exists x{:}T \bullet R \equiv True)$ (x not in P)
$\exists/\wedge$:	$(\exists x{:}T \mid R \bullet P \wedge Q) \Rightarrow (\exists x{:}T \mid R \bullet P) \wedge (\exists x{:}T \mid R \bullet Q)$
$\wedge/\exists$:	$P \wedge (\exists x{:}T \mid R \bullet Q) \equiv (\exists x{:}T \mid R \bullet P \wedge Q)$ (x not in P)
$\exists$-lub:	$(\exists x{:}T \mid R \bullet P) \Rightarrow Q \equiv (\forall x{:}T \mid R \bullet P \Rightarrow Q)$ (x not in Q)
$\Rightarrow/\exists$:	$P \Rightarrow (\exists x{:}T \mid R \bullet Q) \equiv (\exists x{:}T \mid R \bullet P \Rightarrow Q)$ if $(\exists x{:}T \bullet R \equiv True)$ (x not in P)
de Morgan:	$\neg(\exists x{:}T \mid R \bullet P) \equiv (\forall x{:}T \mid R \bullet \neg P)$
$\Delta\exists$:	$(\forall x{:}T \bullet \Delta P) \Rightarrow \Delta(\exists x{:}T \bullet P)$

Monotonicity of quantifications

$\forall/\equiv$:	$(\forall x{:}T \mid R \bullet P \equiv Q) \Rightarrow ((\forall x{:}T \mid R \bullet P) \equiv (\forall x{:}T \mid R \bullet Q))$
$\forall/\Rightarrow$:	$(\forall x{:}T \mid R \bullet P \Rightarrow Q) \Rightarrow ((\forall x{:}T \mid R \bullet P) \Rightarrow (\forall x{:}T \mid R \bullet Q))$
$\forall$ range anti-mono:	$(\forall x{:}T \bullet R \Rightarrow S) \Rightarrow ((\forall x{:}T \mid S \bullet P) \Rightarrow (\forall x{:}T \mid R \bullet P))$
$\exists/\equiv$:	$(\forall x{:}T \mid R \bullet P \equiv Q) \Rightarrow ((\exists x{:}T \mid R \bullet P) \equiv (\exists x{:}T \mid R \bullet Q))$
$\exists/\Rightarrow$:	$(\forall x{:}T \mid R \bullet P \Rightarrow Q) \Rightarrow ((\exists x{:}T \mid R \bullet P) \Rightarrow (\exists x{:}T \mid R \bullet Q))$
$\exists$ range mono:	$(\forall x{:}T \bullet S \Rightarrow R) \Rightarrow ((\exists x{:}T \mid S \bullet P) \Rightarrow (\exists x{:}T \mid R \bullet P))$

Truth and falsity in quantifications

$\forall$-truth:	$((\forall x{:}T \mid R \bullet P) \equiv True) \equiv (\forall x{:}T \mid R \bullet P \equiv True)$
$\forall$-falsity:	$((\forall x{:}T \mid R \bullet P) \equiv False) \equiv (\exists x{:}T \mid R \bullet P \equiv False)$
$\forall$-non-truth:	$((\forall x{:}T \mid R \bullet P) \not\equiv True) \equiv (\exists x{:}T \mid R \bullet P \not\equiv True)$
$\exists$-truth:	$((\exists x{:}T \mid R \bullet P) \equiv True) \equiv (\exists x{:}T \mid R \bullet P \equiv True)$
$\exists$-falsity:	$((\exists x{:}T \mid R \bullet P) \equiv False) \equiv (\forall x{:}T \mid R \bullet P \equiv False)$
$\exists$-non-truth:	$((\exists x{:}T \mid R \bullet P) \not\equiv True) \equiv (\forall x{:}T \mid R \bullet P \not\equiv True)$

Constant predicates

∀-idem: $(\forall x{:}T \mid R \bullet \mathrm{True})$

∀-constant: $(\forall x{:}T \mid R \bullet P) \equiv P \vee (\forall x{:}T \bullet R \not\equiv \mathrm{True})$ if x not in P.

∀-idem: $(\forall x{:}T \bullet P) \equiv P$ if x not in P.

habitation: $(\forall x{:}T \mid R \bullet \mathrm{False}) \equiv (\forall x{:}T \bullet R \not\equiv \mathrm{True})$

habitation: $\neg(\forall x{:}T \bullet \mathrm{False})$

∃-idem: $\neg(\exists x{:}T \mid R \bullet \mathrm{False})$

∃-constant: $(\exists x{:}T \mid R \bullet P) \equiv P \wedge (\exists x{:}T \bullet R \equiv \mathrm{True})$ if x not in P.

∃-idem: $(\exists x{:}T \bullet P) \equiv P$ if x not in P.

habitation: $(\exists x{:}T \mid R \bullet \mathrm{True}) \equiv (\exists x{:}T \bullet R \equiv \mathrm{True})$

habitation: $(\exists x{:}T \bullet \mathrm{True})$

Instantiation

instantiation: $(\forall x{:}T \bullet P) \Rightarrow P$

instantiation: $(\forall x{:}T \bullet P) \Rightarrow P[x{:=}t]$ if Δt

instantiation: $(\forall x{:}T \bullet P) \wedge P[x{:=}t] \equiv (\forall x{:}T \bullet P)$ if Δt

instantiation: $(\forall x{:}T \bullet P) \vee P[x{:=}t] \equiv P[x{:=}t]$ if Δt

instantiation: $(\forall x{:}T \mid R \bullet P) \Rightarrow (R \Rightarrow P)$

𝔹-instantiation: $(\forall x{:}\mathbb{B} \bullet P) \equiv P[x{:=}\mathrm{True}] \wedge P[x{:=}\mathrm{False}]$

instantiation: $P \Rightarrow (\exists x{:}T \bullet P)$

instantiation: $P[x{:=}t] \Rightarrow (\exists x{:}T \bullet P)$ if Δt

instantiation: $(\exists x{:}T \bullet P) \vee P[x{:=}t] \equiv (\exists x{:}T \bullet P)$ if Δt

instantiation: $(\exists x{:}T \bullet P) \wedge P[x{:=}t] \equiv P[x{:=}t]$ if Δt

instantiation: $R \wedge P \Rightarrow (\exists x{:}T \mid R \bullet P)$

𝔹-instantiation: $(\exists x{:}\mathbb{B} \bullet P) \equiv P[x{:=}\mathrm{True}] \vee P[x{:=}\mathrm{False}]$

Dummy manipulation

renaming: $(\forall x{:}T \bullet P) \equiv (\forall y{:}T \bullet P[x{:=}y])$ where y is fresh

interchange: $(\forall x{:}T \mid R \bullet (\forall y{:}U \mid S \bullet P)) \equiv (\forall y{:}U \mid S \bullet (\forall x{:}T \mid R \bullet P))$ if x not in S, y not in R.

renaming: $(\exists x{:}T \bullet P) \equiv (\exists y{:}T \bullet P[x{:=}y])$ where y not in P.

interchange: $((\exists x{:}T \mid R \bullet (\exists y{:}U \mid S \bullet P)) \equiv ((\exists y{:}U \mid S \bullet (\exists x{:}T \mid R \bullet P))$ if x not in S, y not in R.

∃/∀: $(\exists x{:}T \mid R \bullet (\forall y{:}U \mid S \bullet P)) \Rightarrow (\forall y{:}U \mid S \bullet (\exists x{:}T \mid R \bullet P))$ if x not in S, y not in R

One-point and shifting

one-point: $(\forall x{:}T \mid x \equiv t \bullet P) \equiv \Delta t \Rightarrow P[x{:=}t]$ where x not in t.

shifting: $(\forall x{:}T \mid R \bullet P) \equiv (\forall x{:}T \mid R[x{:=}t] \bullet P[x{:=}t])$ if t surjective and total in x

one-point: $(\exists x{:}T \mid x \equiv t \bullet P) \equiv P[x{:=}t] \wedge \Delta t$ where x not in t.

shifting: $(\exists x{:}T \mid R \bullet P) \equiv (\exists x{:}T \mid R[x{:=}t] \bullet P[x{:=}t])$ if t surjective and total in x

Miscellaneous

$\forall\wedge\exists$:	$(\forall x{:}T \mid R \bullet P) \wedge (\exists x{:}T \mid R \bullet Q) \Rightarrow (\exists x{:}T \mid R \bullet P \wedge Q)$
$\forall$-$\exists$:	$(\forall x{:}T \mid R \bullet P) \Rightarrow (\exists x{:}T \mid R \bullet P)$ if $(\exists x{:}T \bullet R \equiv \text{True})$
$\forall$ strengthen:	$(\forall x{:}T \bullet Q) \Rightarrow ((\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet Q \Rightarrow P))$

Refinement

$\sqsubseteq$-defn:	$E \sqsubseteq F \equiv (\forall x{:}T \bullet F \sqsubseteq x \Rightarrow E \sqsubseteq x)$ x fresh
$\sqsubseteq$-antisymm:	$E \sqsubseteq F \wedge F \sqsubseteq E \Rightarrow E \equiv F$
no-miracles:	$(\exists x{:}T \bullet E \sqsubseteq x)$ where E has type T and x fresh
$\mathbb{B}$-flat:	$(\forall x,y{:}\mathbb{B} \bullet x \sqsubseteq y \equiv (x \equiv y))$
$\bot\sqsubseteq$:	$\bot \sqsubseteq E$

Equality

=-defn:	$E{=}F \sqsubseteq \text{True} \equiv (\exists x{:}T \mid E \sqsubseteq x \bullet (\exists y{:}T \mid F \sqsubseteq y \bullet x \equiv y))$ x,y fresh
=-defn:	$E{=}F \sqsubseteq \text{False} \equiv \neg\Delta E \vee \neg\Delta F \vee (\exists x{:}T \mid E \sqsubseteq x \bullet (\exists y{:}T \mid F \sqsubseteq y \bullet x \not\equiv y))$ x,y fresh
=-truth:	$(E{=}F \equiv \text{True}) \equiv \nabla E \wedge (E \equiv F)$
∇-defn:	$\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x))$ x fresh
$\nabla\mathbb{B}$:	$\nabla P \equiv \Delta P$
=-symm:	$E{=}F \equiv F{=}E$
=-refl:	$\nabla E \Rightarrow E{=}E$
=-trans:	$E{=}F \wedge F{=}G \Rightarrow E{=}G$
$=\bot$:	$E{=}\bot \equiv \bot$
=-$\equiv$:	$E{=}F \Rightarrow (E \equiv F)$
$=\sqsubseteq$-mono:	$E \sqsubseteq F \Rightarrow (E{=}G) \sqsubseteq (F{=}G)$ and $F \sqsubseteq G \Rightarrow (E{=}F) \sqsubseteq (E{=}G)$
=-$\mathbb{B}$:	$P{=}Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$