Many-valued Logics for Programming: Theorems and Proofs

Joseph M. Morris and Alexander Bunkenburg

e-mail: jmm@dcs.glasgow.ac.uk

UNIVERSITY of GLASGOW

REPORT R-1999-49

Dept of Computing Science
University of Glasgow
Glasgow G12 8QQ
Scotland, UK

Version: 990907

TABLE OF CONTENTS

1. Introduction

1.1 A family of many-valued logics

E3, **EC**, **EB**, and **E4** are members of a family of many-valued logics for reasoning equationally about programs and specifications.

- **E3** [5] is a three-valued logic designed to handle partiality; it is more or less an equational version of typed LPF ("logic of partial functions") [3,4].
- **EC** (which we formally called **E□**) is also three-valued, but designed to accommodate nondeterminacy rather than partiality; it is described in [5].
- **E4** is a four-valued logic that accommodates both partiality and nondeterminacy [6].
- **EB** is also four-valued, and supports nondeterminacy and miracles (i.e. empty choices); it is designed to support bunch theory and is described in [7].

They all share a core set of axioms and inference rules, which we call $\mathbf{E}^{-}$. The father of this family is **E** [1,2] which is a version of traditional predicate calculus formulated by Dijkstra and Feijen to facilitate an algebraic style of constructive proving. The name "**E**" is taken from the first letter in "equational", because the most important connective in the logic is equivalence rather than implication.

Here we prove the important theorems of the logics, and describe and justify the proof techniques that allow them to be used effectively in proof engineering. The paper is a reference manual, and is not intended to be read through. It is written partly as a definitive record of the various logics in the family, and partly in response to the requests and challenges we have received to prove this or that theorem.

In bringing all the logics under one roof, we have taken the opportunity to improve the axiomatisations in various ways. In particular, some axioms special to **E4** turn out to have been derivable, and the treatment of equality has been overhauled to make it more uniform across all members of the family. In our original presentation of **E4**, we erroneously stated that conjuntion distributes over disjunction (and vice versa), but this is not in fact an equivalence but a bi-implication; this is corrected here.

Our interest is in reasoning about strongly typed specification and programming languages. Typed specification and programming languages are axiomatised by providing a collection of axioms for each type, together with a set of inference rules. We assume that the booleans are among the types, and that they include universal and existential quantifications. By a "logic" we mean an axiomatisation of the boolean type, together with the inference rules. Relationships among programs, such as equivalence or refinement, may be treated as boolean terms, with the result that for everyday reasoning about programs, no extra logic is needed at the metalevel.

1.2 Overview and notation

We use the letters E, F, G, ..., and t to stand for terms of arbitrary type. The letters T, U, ... stand for arbitrary types (including the booleans). The type symbol $\mathbb{B}$ (pronounced "bool") denotes the booleans. Terms of type $\mathbb{B}$ are called "formulae"; we let P, Q, and R stand for formulae. We are also given for each type T an infinite supply of program variables. We let $\mathbf{x}_T$ stand for an arbitrary program variable of type T; in programming languages, the type is supplied by context and so the subscript is typically omitted. For x a variable of any kind, we denote by E[x:=t] the term got by substituting each free occurrence of x in E with t (with renaming as necessary to avoid free variables in t becoming bound as a result of the substitution). We are given two boolean constants, namely True and False; these constants and the program variables of type $\mathbb{B}$ together constitute the "atomic" formulae. Other types supply further atomic formulae (for example, the integers provide 2<3), but at this level of presentation we do not care about the details of other types. We construct complex formulae in the usual way: $P \wedge Q$, $P \vee Q$, $P \Rightarrow Q$, $P \Leftrightarrow Q$, $E \equiv F$, $E \not\equiv F$, $\neg P$, $(\exists x{:}T \bullet P)$, $(\forall x{:}T \bullet P)$, $(\exists x{:}T|R \bullet P)$, and $(\forall x{:}T|R \bullet P)$.

Classical logics, including **E**, assume that typed terms denote a single unique value in the type. In programming and specification languages, however, this is not necessarily so. A non-terminating invocation of a recursive function of result type T does not yield a value in T; we say that its outcome is the special value $\perp_T$ ($\perp$ is pronounced "bottom"). Specification languages may include terms of the form $E \sqcap F$ to denote a nondeterministic choice among terms E and F. An evaluation of $E \sqcap F$ may yield different outcomes on different occasions, the only requirement being that each outcome is chosen from an evaluation of E or an evaluation of F. Specification languages may also provide a mechanism for constructing a term which denotes a value satisfying some defining property. If it turns out that no value satisfies the property then the term has no outcomes and we treat it as a special term called $null_T$ where T denotes its type. These special terms may even appear in the booleans, giving us strange booleans such as $\perp_{\mathbb{B}}$, $True \sqcap False$, and $null_{\mathbb{B}}$. **E3** supports $\perp_T$; **EC** supports $\sqcap$; **EB** supports $\sqcap$ and $null_T$; and **E4** supports $\perp_T$ and $\sqcap$.

We introduce the prefix symbol Δ where ΔE is boolean-valued. The intention is that ΔE holds just when E denotes a single unique value in the traditional sense. For example, $\Delta(3*2)$ holds, but not $\Delta(3/0)$ or $\Delta(True \sqcap False)$. A term E satisfying ΔE is said to be "proper"; other terms are said to be "improper". The core logic. $\mathbf{E}^{-}$ axiomatises the boolean operators so that they retain their traditional meaning when all the arguments are proper, but it leaves open the behaviour of the operators with respect to the strange booleans. However, it does establish that some terms are always proper, most notably $E \equiv F$, ΔE, and program variables. From the axioms of $\mathbf{E}^{-}$ we can deduce that there are two distinct booleans (True and False), but we cannot conclude that there are not more. Despite this openness, it turns out that nearly all the workhorse theorems of classical predicate logic are derivable in $\mathbf{E}^{-}$. We subsequently construct each logic in the family by (in each case) adding a small collection of further axioms. For example, **E** is just $\mathbf{E}^{-}$ plus axiom ΔE.

> Theorems that do not pertain to all logics (i.e. those that rely on axioms not in $\mathbf{E}^{-}$) are labelled with superscripts as a reminder of the logics to which they belong. In the superscripts 2=**E**, 3=**E3**, c=**EC**, b=**EB**, and 4=**E4**. Axioms are labelled similarly.

The underlying model is briefly as follows. Disjunction and conjunction will be formalised so that they behave as minimisation operations with respect to the following orderings:

Disjunction:	$True < null_{\mathbb{B}} < \perp_{\mathbb{B}} < True \sqcap False < False$
Conjunction:	$False < null_{\mathbb{B}} < \perp_{\mathbb{B}} < True \sqcap False < True$

Because $\perp_{\mathbb{B}}$ and $null_{\mathbb{B}}$ do not appear together in any logic, their relative order above is irrelevant. Of course, $\mathbf{E}^{-}$ fixes only the relative ordering of True and False, and has nothing to say of other boolean values; their place is fixed in the respective logics. $\mathbf{E}^{-}$ fixes negation so that it behaves classically with respect to the proper booleans, and otherwise is an involution; subsequently, each particular logic extends negation to the strange booleans by treating it as an identity operation on them. The implication P⇒Q is treated as being equivalent to (P≢True)∨Q. Equivalence (≡) is so-called "strong equivalence", i.e it yields True if its two arguments are the same, and otherwise it yields False. We will introduce equality (=) later. We will formalise (∀x:T • P) so that it behaves as a generalised conjunction of the formulae obtained by instantiating P with the *proper* elements of T. Existential quantification is treated similarly as a generalised disjunction (and indeed we will retain the equivalence of (∃x:T • P) and ¬(∀x:T • ¬P)). Quantifications over subtypes are written as (∀x:T|R • P) and (∃x:T|R • P), respectively, where in each case R is called the "range". We will formalise (∀x:T|R • P) so that it encodes "P for all x for which R is True", and (∃x:T|R • P) so that it encodes "P for some x that for which R is True".

Brackets may be omitted using the following operator precedence, which runs from highest to lowest (the list includes τ and ⊑ which we will explain later):

Δ, τ, ¬   =   ⊓   ⊑   ∧, ∨   ⇒, ⇔   ≡, ≢

Operators in comma-separated groups have the same priority. Prefix unary operators such as ¬ and Δ bracket to the right. Substitution binds tightest of all.

2. The core logic E‾

2.1 Axioms and inference rules of E‾

The axioms common to all the logics are all substitution instances of the following formulae:

Equivalence

≡-refl:	E ≡ E
≡-symm:	(E ≡ F) ≡ (F ≡ E)
≡-truth:	((E ≡ F) ≡ True) ≡ (E ≡ F)

Negation

exchange:	(¬P ≡ Q) ≡ (¬Q ≡ P)
≢-defn:	(E ≢ F) ≡ ¬(E ≡ F)
False-defn:	False ≡ ¬True

Disjunction

∨-symm:	P ∨ Q ≡ Q ∨ P
∨-assoc:	P ∨ (Q ∨ R) ≡ (P ∨ Q) ∨ R
∨-idem:	P ∨ P ≡ P
∨-zero:	P ∨ True ≡ True
∨-unit:	P ∨ False ≡ P

Conjunction

∧-defn:	$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$

Implication

⇒-defn:	$P \Rightarrow Q \equiv (P \not\equiv True) \vee Q$
⇒/≡:	$P \Rightarrow (Q \equiv R) \equiv (P \Rightarrow Q \equiv P \Rightarrow R)$
⇒/∧:	$P \Rightarrow Q \wedge R \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$
∨-lub:	$P \vee Q \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$
shunting:	$P \wedge Q \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$
≡-weakening:	$(P \equiv Q) \Rightarrow (P \Rightarrow Q)$
Leibniz:	$(E \equiv F) \Rightarrow (G[x:=E] \equiv G[x:=F])$
⇔-defn:	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Boolean properness

$\Delta\mathbb{B}$:	$\Delta P \equiv ((P \equiv True) \equiv P)$

Universal quantification

∀/∧:	$(\forall x{:}T \bullet P \wedge Q) \;\equiv\; (\forall x{:}T \bullet P) \wedge (\forall x{:}T \bullet Q)$
⇒/∀:	$(\forall x{:}T \bullet P \Rightarrow Q) \;\equiv\; P \Rightarrow (\forall x{:}T \bullet Q)$ if x does not occur free in P.
∀/≡:	$(\forall x{:}T \bullet P \equiv Q) \Rightarrow ((\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet Q))$
∀-truth:	$((\forall x{:}T \bullet P) \equiv True) \;\equiv\; (\forall x{:}T \bullet P \equiv True)$
interchange:	$(\forall x{:}T \bullet (\forall y{:}U \bullet P)) \;\equiv\; (\forall y{:}U \bullet (\forall x{:}T \bullet P))$
renaming:	$(\forall x{:}T \bullet P) \;\equiv\; (\forall y{:}T \bullet P[x:=y])$ where y is fresh
trading:	$(\forall x{:}T \vert R \bullet P) \;\equiv\; (\forall x{:}T \bullet R \Rightarrow P)$

Existential quantification

∃-defn:	$(\exists x{:}T \bullet P) \;\equiv\; \neg(\forall x{:}T \bullet \neg P)$
∃|-defn:	$(\exists x{:}T \vert R \bullet P) \;\equiv\; \neg(\forall x{:}T \vert R \bullet \neg P)$
∃-lub:	$(\exists x{:}T \bullet P) \Rightarrow Q \;\equiv\; (\forall x{:}T \bullet P \Rightarrow Q)$ if x does not occur free in Q.

Term properness

instantiation:	$(\forall x{:}T \bullet P) \wedge \Delta t \;\Rightarrow\; P[x:=t]$
variables proper:	$\Delta \mathbf{x}_T$
constants proper:	$(\forall x{:}T \bullet \Delta x)$

∧-defn, ∃-defn, and ∃|-defn are all instances of de Morgan's laws, and we sometimes refer to them as such in proofs.

The axioms differ from those of **E** in that that $(P \equiv True) \equiv P$ holds only for proper P, and in that instantiation is only valid for proper terms.

The inference rules are modus ponens (MP) and generalisation:

Modus Ponens $$\frac{P,\ P \Rightarrow Q}{Q}$$

Generalisation $$\frac{P}{(\forall x{:}T \bullet P)}$$

2.2 Derived inference rules

The archetypical proof in the logic family consists of transforming the formula to be proved into a known theorem. The transformation consists of repeatedly substituting equals for equals. This method relies on a collection of derived inference rules, namely Equanimity, Leibniz, Transitivity, True-intro, False-intro, and Implication Transitivity. We prove the validity of each in turn, and then explain how they are used (Implication Transitivity is not needed immediately and we postpone its verification until later).

Equational reasoning proceeds using the following inference rules:

Equanimity $$\frac{P,\ P \equiv Q}{Q}$$

Leibniz $$\frac{E \equiv F,\ P[x{:=}E]}{P[x{:=}F]}$$

Transitivity $$\frac{E \equiv F,\ F \equiv G}{E \equiv G}$$

True-intro $$\frac{P}{P \equiv \text{True}}$$

False-intro $$\frac{\neg P}{P \equiv \text{False}}$$

The symmetry of $\equiv$ gives rise to trivial variations on each of the above.

We prove the validity of each in turn by showing a sequence of steps by which the conclusion (the formula below the line) can be deduced from the hypotheses (the formulae above the line).

Equanimity

(i)	$P \equiv Q$	— hypothesis
(ii)	$(P \equiv Q) \Rightarrow (P \Rightarrow Q)$	— $\equiv$-weakening
(iii)	$P \Rightarrow Q$	— (i), (ii), MP
(iv)	P	— hypothesis
(v)	Q	— (iv), (iii), MP

Leibniz

(i)	$E \equiv F$	— hypothesis
(ii)	$(E \equiv F) \Rightarrow (P[x{:=}E] \equiv P[x{:=}F])$	— axiom Leibniz
(iii)	$P[x{:=}E] \equiv P[x{:=}F]$	— (i), (ii), MP
(iv)	$P[x{:=}E]$	— hypothesis
(v)	$P[x{:=}F]$	— (iv), (iii), Equanimity

Transitivity

(i)	$E \equiv F$	— hypothesis
(ii)	$(E \equiv F) \equiv (F \equiv E)$	— $\equiv$-symm

(iii)	F ≡ E	— (i), (ii), Equanimity
(iv)	F ≡ G	— hypothesis
(v)	E ≡ G	— (iii), (iv), Leibniz

True-intro
We shall require the theorem (P ≡ True) ∨ (P ≢ True); this is proved as follows:

(a)	(P ≡ True) ⇒ (P ≡ True)	— axiom Leibniz with x for G
(b)	(a) ≡ ((P ≡ True) ≢ True) ∨ (P ≡ True)	— ⇒-defn
(c)	((P ≡ True) ≢ True) ∨ (P ≡ True)	— (a), (b), Equanimity
(d)	((P ≡ True) ≢ True) ≡ (P ≢ True)	— theorem ≡-non-truth (see below)
(e)	(P ≢ True) ∨ (P ≡ True)	— (c), (d), Leibniz

The justification of True-intro is:

(i)	P	— hypothesis
(ii)	P ⇒ (P ≡ True) ≡ (P≢True) ∨ (P ≡ True)	— ⇒-defn
(iii)	P ⇒ (P ≡ True)	— (e), (ii), Equanimity
(iv)	P ≡ True	— (i), (iii), MP

Theorem ≡-non-truth is ((E ≡ F) ≢ True) ≡ (E ≢ F); it is easily proved using ≢-defn and ≡-truth, without use of True-intro.

2.3 Reasoning with equivalence

A proof that P is a theorem is typically laid out as follows:

```
        P
≡"justification 1"
        Q
≡"justification 2"
        R              — Theorem "T"
```

This is short-hand for:

(i)	P ≡ Q	— "justification 1"
(ii)	Q ≡ R	— "justification 2"
(iii)	P ≡ R	— (i), (ii), Transitivity
(iv)	R	— Theorem "T"
(v)	P	— (iii), (iv), Equanimity

In each proof step, the justification of X ≡ Y (for any X and Y) takes one of the following forms.

1. It may be reference to where X ≡ Y has been established as a theorem.
2. If X and Y are similar in structure except that X has a subexpression E where Y has subexpression F, then X ≡ Y can be cast in the form Z[x:=E] ≡ Z[x:=F] where x stands for a fresh variable. In that

case, the justification consists of a reference to where $E \equiv F$ has been established as a theorem. The truth of $Z[x{:=}E] \equiv Z[x{:=}F]$ follows from axiom Leibniz and an application of MP.

3. If $X \equiv Y$ can be cast in the form $Z[x{:=}P] \equiv Z[x{:=}True]$, its justification consists of a reference to where P has been established as a theorem. The conclusion follows as before, with an additional appeal to True-intro to infer $P \equiv True$.
4. If $X \equiv Y$ can be cast in the form $Z[x{:=}P] \equiv Z[x{:=}False]$, the justification consists of a reference to where $\neg P$ has been established as a theorem. The conclusion follows as above, except we appeal to False-intro.
5. Occasionally, a reference to where $E \equiv F$ has been established is packaged as a pair of references, one to where $P \Rightarrow (E \equiv F)$ has been established for some P, and another to where P has been established; $E \equiv F$ follows from an application of MP.

The foregoing describes a proof carried out in two steps; the generalisation to any number of steps is obvious.

Comparing the proof presentation above with its first expansion (i.e. (i) to (v)), we see that step (iii) establishes $P \equiv R$. It follows that we may prove $P \equiv R$ using just this proof presentation, but without a justification of R in the final line. In short, we may prove an equivalence by reducing one side to the other.

3. Propositional E^{-}

3.1 Negation

Theorem ¬-exchange: $(\neg P \equiv Q) \equiv (P \equiv \neg Q)$

Proof:

$\neg P \equiv Q$
≡"exchange"
$\neg Q \equiv P$
≡"≡-symm"
$P \equiv \neg Q$ ❑

Theorem ¬-inv: $\neg\neg P \equiv P$

Proof:

$\neg\neg P \equiv P$
≡"¬-exchange"
$\neg P \equiv \neg P$— ≡-symm ❑

Theorem ≢-symm: $(E \not\equiv F) \equiv (F \not\equiv E)$

Proof:

$E \not\equiv F$
≡"≢-defn"
$\neg(E \equiv F)$
≡"≡-symm"
$\neg(F \equiv E)$
≡"≢-defn"
$F \not\equiv E$ ❑

Theorem ≡-mirror : $(P \equiv Q) \equiv (\neg P \equiv \neg Q)$

Proof:

$(\neg P \equiv \neg Q)$

≡"¬-exchange"

$(\neg\neg P \equiv Q)$

≡"¬-inv"

$(P \equiv Q)$ ❑

Theorem ≡-non-truth: $((E \equiv F) \not\equiv \text{True}) \equiv (E \not\equiv F)$

Proof:

$(E \equiv F) \not\equiv \text{True}$

≡"≢-defn"

$\neg((E \equiv F) \equiv \text{True}))$

≡"≡-truth"

$\neg(E \equiv F)$

≡"≢-defn"

$E \not\equiv F$ ❑

3.2 Disjunction and conjunction

Theorem de Morgan (i) $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
(ii) $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Proof:

(i) $\neg(P \wedge Q)$

≡"∧-defn"

$\neg(\neg(\neg P \vee \neg Q))$

≡"¬-inv"

$(\neg P \vee \neg Q)$

(ii) $\neg P \wedge \neg Q$

≡"∧-defn"

$\neg(\neg\neg P \vee \neg\neg Q)$

≡"¬-inv twice"

$\neg(P \vee Q)$ ❑

Theorem ∧-symm: $(P \wedge Q) \equiv (Q \wedge P)$

Proof

$(P \wedge Q)$

≡"∧-defn"

$\neg(\neg P \vee \neg Q)$

≡"∨-symm"

$\neg(\neg Q \vee \neg P)$

≡"∧-defn"

$(Q \wedge P)$ ❑

Theorem ∧-assoc: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

Proof

$P \wedge (Q \wedge R)$

≡"∧-defn"

$P \wedge \neg(\neg Q \vee \neg R)$

≡"∧-defn"

$\neg(\neg P \vee \neg\neg(\neg Q \vee \neg R))$

≡"¬-inv"

$\neg(\neg P \vee (\neg Q \vee \neg R))$

≡"∨-assoc"

$\neg((\neg P \vee \neg Q) \vee \neg R)$

≡"de Morgan"

$\neg(\neg(P \wedge Q) \vee \neg R)$

≡"∧-defn"

$(P \wedge Q) \wedge R$ ❑

Theorem ∧-idem: $P \wedge P \equiv P$

Proof:

$(P \wedge P)$

≡"∧-defn"

$\neg(\neg P \vee \neg P)$

≡"∨-idem"

$\neg(\neg P)$

≡"¬-inv"

P ❑

Theorem ∨/∨: $P \vee (Q \vee R) \equiv (P \vee Q) \vee (P \vee R)$

Proof:

$P \vee (Q \vee R)$

≡"∨-idem"

$(P \vee P) \vee (Q \vee R)$

≡"∨-symm and associativity"

$(P \vee Q) \vee (P \vee R)$ ❑

Theorem ∧/∧: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$

Proof:

$P \wedge (Q \wedge R)$

≡"∧-idem"

$(P \wedge P) \wedge (Q \wedge R)$

≡"∧-assoc and symmetry"

$(P \wedge (Q \wedge R)) \wedge P$

≡"∧-assoc twice"

$(P \wedge Q) \wedge (R \wedge P)$

≡"∧-symm"

$(P \wedge Q) \wedge (P \wedge R)$ ❑

3.3 Substitution

Theorem ⇒-subst: (E ≡ F) ⇒ P[x:=E] ≡ (E ≡ F) ⇒ P[x:=F]

Proof:

(E ≡ F) ⇒ P[x:=E] ≡ (E ≡ F) ⇒ P[x:=F]

≡"⇒/≡"

(E ≡ F) ⇒ (P[x:=E] ≡ P[x:=F]) — Leibniz ❑

Theorem ∧-subst: (E ≡ F) ∧ P[x:=E] ≡ (E ≡ F) ∧ P[x:=F]

Proof:

(E ≡ F) ∧ P[x:=E] ≡ (E ≡ F) ∧ P[x:=F]

≡"≡-mirror"

¬((E ≡ F) ∧ P[x:=E]) ≡ ¬((E ≡ F) ∧ P[x:=F])

≡"de Morgan twice, ≢-defn twice"

((E ≢ F) ∨ ¬P[x:=E]) ≡ ((E ≢ F) ∨ ¬P[x:=F])

≡"≡-non-truth twice"

((E ≡ F) ≢ True) ∨ ¬P[x:=E]) ≡ ((E ≡ F) ≢ True) ∨ ¬P[x:=F])

≡"⇒-defn twice"

((E ≡ F) ⇒ ¬P[x:=E]) ≡ ((E ≡ F) ⇒ ¬P[x:=F]) — ⇒-subst ❑

Theorem ∨-subst: (E ≢ F) ∨ P[x:=E] ≡ (E ≢ F) ∨ P[x:=F]

3.4 True and False and derived inference rule True-elim

Theorem: True

Proof:

(i) True ≡ True — ≡-refl

(ii) (True ≡ True) ≡ True — True-intro

(iii) True — (i), (ii), Equanimity ❑

Theorem negating False: ¬False

Theorem ∨-zero : True ∨ P ≡ True

Proof: Axiom ∨-zero and ∨-symm. ❑

Theorem ∨-unit : False ∨ P ≡ P

Theorem ∧-zero : (i) P ∧ False ≡ False

(ii) False ∧ P ≡ False

Proof:

P ∧ False ≡ False

≡"≡-mirror, de Morgan"

¬P ∨ ¬False ≡ ¬False

≡"negating False twice"

¬P ∨ True ≡ True — ∨-zero ❑

Theorem ∧-unit: (i) P ∧ True ≡ P

(ii) $\text{True} \wedge P \equiv P$

Theorem ∨-truth: $(P \vee Q \equiv \text{True}) \equiv (P \equiv \text{True}) \vee (Q \equiv \text{True})$

Proof:

$(P \vee Q) \equiv \text{True}$

≡"¬-inv, ≢-defn"

$\neg(P \vee Q \not\equiv \text{True})$

≡"∨-unit"

$\neg((P \vee Q \not\equiv \text{True}) \vee \text{False})$

≡"⇒-defn"

$\neg(P \vee Q \Rightarrow \text{False})$

≡"∨-lub"

$\neg((P \Rightarrow \text{False}) \wedge (Q \Rightarrow \text{False}))$

≡"⇒-defn (twice), ∨-unit (twice)"

$\neg((P \not\equiv \text{True}) \wedge (Q \not\equiv \text{True}))$

≡"de Morgan, ≢-defn (twice), ¬-inv (twice)"

$(P \equiv \text{True}) \vee (Q \equiv \text{True})$ ❑

Theorem ∧-truth: $(P \wedge Q \equiv \text{True}) \equiv (P \equiv \text{True}) \wedge (Q \equiv \text{True})$

Proof:

True

≡"shunting"

$P \wedge Q \Rightarrow \text{False} \equiv P \Rightarrow (Q \Rightarrow \text{False})$

≡"⇒-defn, ∨-unit"

$(P \wedge Q \not\equiv \text{True}) \equiv (P \not\equiv \text{True}) \vee (Q \not\equiv \text{True})$

≡"≡-mirror, de Morgan"

$(P \wedge Q \equiv \text{True}) \equiv (P \equiv \text{True}) \wedge (Q \equiv \text{True})$ ❑

Theorem ∧-falsity: $(P \wedge Q \equiv \text{False}) \equiv (P \equiv \text{False}) \vee (Q \equiv \text{False})$

Proof:

$(P \wedge Q) \equiv \text{False}$

≡"≡-mirror"

$\neg(P \wedge Q) \equiv \neg\text{False}$

≡"de Morgan, negating False"

$\neg P \vee \neg Q \equiv \text{True}$

≡"∨-truth"

$(\neg P \equiv \text{True}) \vee (\neg Q \equiv \text{True})$

≡"¬-exchange, False-defn twice"

$(P \equiv \text{False}) \vee (Q \equiv \text{False})$ ❑

Theorem ∨-falsity: $(P \vee Q \equiv \text{False}) \equiv (P \equiv \text{False}) \wedge (Q \equiv \text{False})$

Theorem ∨-¬truth: $(P \vee Q \not\equiv \text{True}) \equiv (P \not\equiv \text{True}) \wedge (Q \not\equiv \text{True})$

Theorem ∧-¬truth: $(P \wedge Q \not\equiv \text{True}) \equiv (P \not\equiv \text{True}) \vee (Q \not\equiv \text{True})$

Theorem $\vee$-$\neg$falsity: $(P \vee Q \not\equiv False) \equiv (P \not\equiv False) \vee (Q \not\equiv False)$

Theorem $\wedge$-$\neg$falsity: $(P \wedge Q \not\equiv False) \equiv (P \not\equiv False) \wedge (Q \not\equiv False)$

Theorem two values: $False \not\equiv True$

Proof:

$False \not\equiv True$
$\equiv$"$\not\equiv$-defn"
$\neg(False \equiv True)$
$\equiv$"$\wedge$-unit"
$\neg((False \equiv True) \wedge True)$
$\equiv$"$\wedge$-subst"
$\neg((False \equiv True) \wedge False)$
$\equiv$"$\wedge$-zero"
$\neg False$ — negating False ❑

Theorem not False: $\neg False \equiv True$

Theorem excluded false: $(P \vee \neg P) \not\equiv False$

Proof:

$(P \vee \neg P) \not\equiv False$
$\equiv$"$\not\equiv$-defn"
$\neg(P \vee \neg P \equiv False)$
$\equiv$"$\vee$-falsity"
$\neg((P \equiv False) \wedge (\neg P \equiv False))$
$\equiv$"de Morgan, $\not\equiv$-defn"
$(P \not\equiv False) \vee (\neg P \not\equiv False)$
$\equiv$"$\vee$-subst"
$(P \not\equiv False) \vee (\neg False \not\equiv False)$
$\equiv$"two values, not False"
$(P \not\equiv False) \vee True$ — $\vee$-zero ❑

The following is a simple but useful derived inference rule, called True-elim:

$$\frac{P \equiv True}{P}$$

Proof: True, $P \equiv True$, and Equanimity. ❑

3.5 Implication

Theorem $\Rightarrow$-refl: $P \Rightarrow P$

Proof:

$P \Rightarrow P$
$\equiv$"$\Rightarrow$-defn"
$(P \not\equiv True) \vee P$

≡"∨-subst"
$(P \not\equiv True) \vee True$ — ∨-zero ❑

Theorem ⇒-connected: $(P \Rightarrow Q) \vee (Q \Rightarrow P)$
Proof:
$(P \Rightarrow Q) \vee (Q \Rightarrow P)$
≡"⇒-defn twice"
$(P \not\equiv True) \vee Q \vee (Q \not\equiv True) \vee P$
≡"∨-subst"
$(P \not\equiv True) \vee Q \vee (Q \not\equiv True) \vee True$ — ∨-zero ❑

Theorem True maximal: $P \Rightarrow True$
Proof:
$P \Rightarrow True$
≡"⇒-defn"
$(P \not\equiv True) \vee True$ — ∨-zero ❑

Theorem False minimal: $False \Rightarrow P$
Proof:
$False \Rightarrow P$
≡"⇒-defn"
$\neg(False \equiv True) \vee P$
≡"two values"
$True \vee P$— ∨-zero ❑

Theorem ⇒-left-unit: $True \Rightarrow P \equiv P$

Theorem ⇒-right-zero: $P \Rightarrow True \equiv True$

Theorem ⇒/∨: $P \Rightarrow Q \vee R \equiv (P \Rightarrow Q) \vee (P \Rightarrow R)$
Proof:
$(P \Rightarrow Q) \vee (P \Rightarrow R)$
≡"⇒-defn"
$((P \not\equiv True) \vee Q) \vee ((P \not\equiv True) \vee R)$
≡"∨-assoc, ∨/∨"
$(P \not\equiv True) \vee (Q \vee R)$
≡"⇒-defn"
$P \Rightarrow Q \vee R$ ❑

Theorem ∧\⇒: $P \wedge Q \Rightarrow R \equiv (P \Rightarrow R) \vee (Q \Rightarrow R)$
Proof:
$P \wedge Q \Rightarrow R$
≡"⇒-defn"
$(P \wedge Q \not\equiv True) \vee R$
≡"∧-¬truth"
$(P \not\equiv True) \vee (Q \not\equiv True) \vee R$

$\equiv$"$\vee$-idem"
$(P \not\equiv True) \vee R \vee (Q \not\equiv True) \vee R$
$\equiv$"$\Rightarrow$-defn (twice)"
$(P \Rightarrow R) \vee (Q \Rightarrow R)$ ❑

Theorem $\Rightarrow/\Rightarrow$: $P \Rightarrow (Q \Rightarrow R) \equiv (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$

Proof:

$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$
$\equiv$"$\Rightarrow$-defn"
$(P \Rightarrow Q \not\equiv True) \vee (P \Rightarrow R)$
$\equiv$"$\Rightarrow$-defn"
$(P \Rightarrow Q \not\equiv True) \vee (P \not\equiv True) \vee R$
$\equiv$"$\vee$-subst"
$(True \Rightarrow Q \not\equiv True) \vee (P \not\equiv True) \vee R$
$\equiv$"$\Rightarrow$-left-unit"
$(Q \not\equiv True) \vee (P \not\equiv True) \vee R$
$\equiv$"$\vee$-symm"
$(P \not\equiv True) \vee (Q \not\equiv True) \vee R$
$\equiv$"$\Rightarrow$-defn (twice)"
$P \Rightarrow (Q \Rightarrow R)$ ❑

Theorem $\Rightarrow/\equiv/\wedge$: $P \Rightarrow (Q \equiv R) \equiv ((P \equiv True) \wedge Q \equiv (P \equiv True) \wedge R)$

Proof:

$P \Rightarrow (Q \equiv R)$
$\equiv$"$\equiv$-mirror"
$P \Rightarrow (\neg Q \equiv \neg R)$
$\equiv$"$\Rightarrow/\equiv$"
$P \Rightarrow \neg Q \equiv P \Rightarrow \neg R$
$\equiv$"$\Rightarrow$-defn"
$\neg(P \equiv True) \vee \neg Q \equiv \neg(P \equiv True) \vee \neg R$
$\equiv$"$\equiv$-mirror, de Morgan"
$(P \equiv True) \wedge Q \equiv (P \equiv True) \wedge R$ ❑

Theorem weakening: (i) $P \Rightarrow P \vee Q$
(ii) $P \wedge Q \Rightarrow P$

Proof:

(i) $P \Rightarrow P \vee Q$
$\equiv$"$\Rightarrow/\vee$"
$(P \Rightarrow P) \vee (P \Rightarrow Q)$
$\equiv$"$\Rightarrow$-refl"
$True \vee (P \Rightarrow Q)$ — $\vee$-zero

(ii) $P \wedge Q \Rightarrow P$
$\equiv$"$\Rightarrow$-defn"
$(P \wedge Q \not\equiv True) \vee P$
$\equiv$"$\wedge$-$\neg$truth"

$(P \not\equiv True) \vee (Q \not\equiv True) \vee P$
≡"∨-subst"
$(P \not\equiv True) \vee (Q \not\equiv True) \vee True$
≡"∨-zero"
True ❑

Theorem ⇒-trans: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

Proof:

$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$
≡"∧-symm, shunting"
$(Q \Rightarrow R) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$
≡"⇒/⇒"
$(Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$
≡"shunting"
$(Q \Rightarrow R) \wedge P \Rightarrow (Q \Rightarrow R)$ — weakening ❑

Theorem ≡-trans: $(E \equiv F) \wedge (F \equiv G) \Rightarrow (E \equiv G)$

Proof:

$(E \equiv F) \wedge (F \equiv G) \Rightarrow (E \equiv G)$
≡"∧-subst"
$(E \equiv G) \wedge (F \equiv G) \Rightarrow (E \equiv G)$ — weakening ❑

Theorem transitivity: (i) $(P \equiv Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$
(ii) $(P \Rightarrow Q) \wedge (Q \equiv R) \Rightarrow (P \Rightarrow R)$

Theorem modus ponens: $P \wedge (P \Rightarrow Q) \Rightarrow Q$

Proof:

$P \wedge (P \Rightarrow Q) \Rightarrow Q$
≡"∧-symm, shunting"
$(P \Rightarrow Q) \Rightarrow (P \Rightarrow Q)$ — ⇒-refl ❑

Theorem True-⇒: $(P \equiv True) \Rightarrow Q \equiv P \Rightarrow Q$

Theorem ⇒-truth: (i) $(P \Rightarrow Q \equiv True) \equiv P \Rightarrow (Q \equiv True)$
(ii) $(P \Rightarrow Q \equiv True) \equiv (P \equiv True) \Rightarrow (Q \equiv True)$

Theorem[*] ∨/∧: $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Proof:

(⇒): $P \vee (Q \wedge R) \Rightarrow (P \vee Q) \wedge (P \vee R)$
≡"⇒/∧"
$(P \vee (Q \wedge R) \Rightarrow P \vee Q) \wedge (P \vee (Q \wedge R) \Rightarrow P \vee R)$
≡"∨-lub (twice)"
$(P \Rightarrow P \vee Q) \wedge (Q \wedge R \Rightarrow P \vee Q) \wedge (P \Rightarrow P \vee R) \wedge (Q \wedge R \Rightarrow P \vee R)$
≡"weakening (twice), ∧-unit (twice)"

---

[*] The bi-implication can be replaced by equivalence in 2- and 3-valued logics. See Section 5.

$(Q \wedge R \Rightarrow P \vee Q) \wedge (Q \wedge R \Rightarrow P \vee R)$
≡"⇒/∨ (twice)"
$((Q \wedge R \Rightarrow P) \vee (Q \wedge R \Rightarrow Q)) \wedge ((Q \wedge R \Rightarrow P) \vee (Q \wedge R \Rightarrow R))$
≡"weakening (twice)"
$((Q \wedge R \Rightarrow P) \vee \text{True}) \wedge ((Q \wedge R \Rightarrow P) \vee \text{True})$
≡"∨-zero (twice), ∧-unit"
True

(⇐): $(P \vee Q) \wedge (P \vee R) \Rightarrow P \vee (Q \wedge R)$
≡"∧\⇒"
$(P \vee Q \Rightarrow P \vee (Q \wedge R)) \vee (P \vee R \Rightarrow P \vee (Q \wedge R))$
≡"∨-lub"
$((P \Rightarrow P \vee (Q \wedge R)) \wedge (Q \Rightarrow P \vee (Q \wedge R))) \vee$
$((P \Rightarrow P \vee (Q \wedge R)) \wedge (R \Rightarrow P \vee (Q \wedge R)))$
≡"weakening, ∧-unit"
$(Q \Rightarrow P \vee (Q \wedge R)) \vee (R \Rightarrow P \vee (Q \wedge R))$
≡"∧\⇒"
$Q \wedge R \Rightarrow P \vee (Q \wedge R)$
≡"weakening"
True ❑

Theorem* ∧/∨: $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
Proof:
(⇒): $P \wedge (Q \vee R) \Rightarrow (P \wedge Q) \vee (P \wedge R)$
≡"⇒-defn, ∧-¬truth"
$(P \not\equiv \text{True}) \vee (Q \vee R \not\equiv \text{True}) \vee (P \wedge Q) \vee (P \wedge R)$
≡"∨-subst (twice)"
$(P \not\equiv \text{True}) \vee (Q \vee R \not\equiv \text{True}) \vee (\text{True} \wedge Q) \vee (\text{True} \wedge R)$
≡"∧-unit (twice)"
$(P \not\equiv \text{True}) \vee (Q \vee R \not\equiv \text{True}) \vee (Q \vee R)$
≡"∨-subst"
$(P \not\equiv \text{True}) \vee (Q \vee R \not\equiv \text{True}) \vee \text{True}$
≡"∨-zero"
True

(⇐): $(P \wedge Q) \vee (P \wedge R) \Rightarrow P \wedge (Q \vee R)$
≡"∨-lub"
$(P \wedge Q \Rightarrow P \wedge (Q \vee R)) \wedge (P \wedge R \Rightarrow P \wedge (Q \vee R))$
≡"⇒/∧"
$(P \wedge Q \Rightarrow P) \wedge (P \wedge Q \Rightarrow Q \vee R) \wedge (P \wedge R \Rightarrow P) \wedge (P \wedge R \Rightarrow Q \vee R)$
≡"weakening, ∧-unit"
$(P \wedge Q \Rightarrow Q \vee R) \wedge (P \wedge R \Rightarrow Q \vee R)$
≡"⇒/∨ (twice)"
$((P \wedge Q \Rightarrow Q) \vee (P \wedge Q \Rightarrow R)) \wedge ((P \wedge R \Rightarrow Q) \vee (P \wedge R \Rightarrow R))$

---

* The bi-implication can be replaced by equivalence in 2- and 3-valued logics. See Section 5.

$\equiv$"weakening (twice)"

(True $\vee$ (P $\wedge$ Q $\Rightarrow$ R)) $\wedge$ ((P $\wedge$ R $\Rightarrow$ Q) $\vee$ True)

$\equiv$"$\vee$-zero (twice), $\wedge$-unit"

True ❑

3.6 Derived inference rules: implication transitivity, $\Rightarrow$-truth, and $\wedge$-intro

A proof that P $\Rightarrow$ Z is a theorem may be conveniently carried out by making use of the following derived inference rule:

Implication Transitivity $\dfrac{P \Rightarrow Q,\ Q \Rightarrow R}{P \Rightarrow R}$

Implication transitivity follows easily from $\Rightarrow$-trans and two applications of MP.
A proof of P $\Rightarrow$ Z may proceed just as described in the preceding section, or it may be laid out as follows:

P

$\equiv$"justification 1"

Q

$\Rightarrow$"justification 2"

R

$\equiv$"justification 3"

Y

$\Rightarrow$"justification 4"

Z

This is shorthand for the following:

(i)	P $\equiv$ Q	— "justification 1"
(ii)	Q $\Rightarrow$ R	— "justification 2"
(iii)	R $\equiv$ Y	— "justification 3"
(iv)	Y $\Rightarrow$ Z	— "justification 4"
(v)	P $\Rightarrow$ R	— (i), (ii), Leibniz
(vi)	P $\Rightarrow$ Y	— (iii), (v), Leibniz
(vii)	P $\Rightarrow$ Z	— (vi), (iv), Implication Transitivity

Justifications of equivalences are as described previously. For justifications of implications we shall need the notion of "positive" and "negative" positions in formulae. A single occurrence of a variable x in a formula Z is said to be "positive" (respectively, "negative") iff Z[x:=P] $\Rightarrow$ Z[x:=Q] follows from P $\Rightarrow$ Q (respectively, Q $\Rightarrow$ P) for all P and Q. A variable x is said to be positive (negative) in Z iff all of its occurrences are positive (negative) in Z. (For example, in x $\Rightarrow$ $\neg$z $\wedge$ (y $\vee$ z), we can show that x is negative, y is positive, and z is neither positive nor negative.) A justification of X $\Rightarrow$ Y takes one of the following forms. It may be a reference to where X $\Rightarrow$ Y has been established as a theorem. Alternatively, if X and Y are similar in structure except that X has sub-formula P where Y has sub-formula Q, then X $\Rightarrow$ Y can be cast in the form Z[x:=P] $\Rightarrow$ Z[x:=Q] where x stands for a fresh variable. In that case, the justification of X $\Rightarrow$ Y takes one of the following forms:

(i) a reference to where $P \Rightarrow Q$ has been established, and a reference to theorem(s) guaranteeing that x is positive in Z.

(ii) a reference to where $Q \Rightarrow P$ has been established, and a reference to theorem(s) guaranteeing that x is negative in Z.

In either case $X \Rightarrow Y$ follows from an application of MP.

The generalisation to proofs of $P \Rightarrow Z$ using any number of steps is obvious. A proof of Q may be presented as a proof of $P \Rightarrow Q$ in the above style, for P any theorem; this is immediately justified by MP. When carrying out such a proof it is often attractive to proceed backwards from Q to P using $\Leftarrow$ ("reverse implication") in place of $\Rightarrow$.

Two useful derived inference rules are

$\wedge$-intro $\dfrac{P,\ Q}{P \wedge Q}$ $\qquad$ $\Rightarrow$-truth $\dfrac{(P \equiv True) \Rightarrow (Q \equiv True)}{P \Rightarrow Q}$

$\wedge$-intro is justified by the theorem $(P \Rightarrow (Q \Rightarrow P \wedge Q)$ (verified by a single application of shunting) and two applications of modus ponens; $\Rightarrow$-truth is justified by theorem $\Rightarrow$-truth and True-elim ❑

3.7 The Deduction Theorem for propositional calculus

We write U, V, ..., W $\vdash$ P to denote that P is a theorem when the formulae U, V, ..., W are additional axioms. The deduction theorem states that if U, V, ..., W, P $\vdash$ Q, then U, V, ..., W $\vdash$ $P \Rightarrow Q$. The standard proof for the propositional calculus relies on three theorems [8]:

(i) $(P \Rightarrow (Q \Rightarrow P))$

(ii) $P \Rightarrow P$

(iii) $(P \Rightarrow (Q \Rightarrow R)) \;\Rightarrow\; ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$

(i) is an immediate consequence of the theorems weakening and shunting .

(ii) is $\Rightarrow$-refl .

(iii) is an immediate consequence of $\Rightarrow/\Rightarrow$ and axiom $\equiv$-weakening.

3.8 Further substitution

Theorem $\Rightarrow$-subst: $P \Rightarrow Q[x:=E] \;\equiv\; P \Rightarrow Q[x:=F]$ if $P \Rightarrow (E \equiv F)$

Proof:

$P \Rightarrow Q[x:=E] \;\equiv\; P \Rightarrow Q[x:=F]$

$\equiv$"$P \Rightarrow (E \equiv F)$, $\Rightarrow/\wedge$"

$P \Rightarrow (E \equiv F) \wedge Q[x:=E] \equiv P \Rightarrow (E \equiv F) \wedge Q[x:=F]$

$\equiv$"$\Rightarrow/\equiv$"

$P \Rightarrow ((E \equiv F) \wedge Q[x:=E] \equiv (E \equiv F) \wedge Q[x:=F])$

$\equiv$"$\wedge$-subst"

$P \Rightarrow True$ — True maximal ❑

Theorem $\wedge$-subst: $(P \equiv True) \wedge Q[x:=E] \;\equiv\; (P \equiv True) \wedge Q[x:=F]$ if $P \Rightarrow (E \equiv F)$

3.9 Monotonicity with respect to ⇒

Theorem $\Rightarrow$-right-mono: $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))$

Proof:

$(P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))$

≡"shunting"

$((P \Rightarrow Q) \wedge (R \Rightarrow P)) \Rightarrow (R \Rightarrow Q)$

≡"$\wedge$-symm, $\Rightarrow$-trans"

True ❑

Theorem $\Rightarrow$-left-anti-mono: $(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$

Proof:

$(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$

≡"shunting"

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

≡"$\Rightarrow$-trans"

True ❑

Theorem $\Rightarrow$-combination:

(i) $(P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \wedge R \Rightarrow Q \wedge S)$

(ii) $(P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \vee R \Rightarrow Q \vee S)$

Proof:

(i) Appealing to the deduction theorem we assume $(P \Rightarrow Q)$ and $(R \Rightarrow S)$:

$(P \wedge R \Rightarrow Q \wedge S)$

≡"$\Rightarrow/\wedge$"

$(P \wedge R \Rightarrow Q) \wedge (P \wedge R \Rightarrow S)$

≡"shunting twice"

$(R \Rightarrow (P \Rightarrow Q)) \wedge (P \Rightarrow (R \Rightarrow S))$

≡"assumptions"

$(R \Rightarrow True) \wedge (P \Rightarrow True)$

≡"True maximal twice, $\wedge$-idem"

True ❑

Theorem $\wedge$-mono: (i) $(P \Rightarrow Q) \Rightarrow (P \wedge R \Rightarrow Q \wedge R)$

(ii) $(P \Rightarrow Q) \Rightarrow (R \wedge P \Rightarrow R \wedge Q)$

Proof:

(i) $(P \wedge R) \Rightarrow (Q \wedge R)$

⇐"$\Rightarrow$-combination"

$(P \Rightarrow Q) \wedge (R \Rightarrow R)$

≡"$\Rightarrow$-refl, $\wedge$-unit"

$P \Rightarrow Q$ ❑

Theorem $\vee$-mono: (i) $(P \Rightarrow Q) \Rightarrow (P \vee R \Rightarrow Q \vee R)$

(ii) $(P \Rightarrow Q) \Rightarrow (R \vee P \Rightarrow R \vee Q)$

Proof:

(i) $(P \vee R) \Rightarrow (Q \vee R)$

$\Leftarrow$"$\Rightarrow$-combination"

$(P \Rightarrow Q) \wedge (R \Rightarrow R)$

$\equiv$"$\Rightarrow$-refl, $\wedge$-unit"

$P \Rightarrow Q$ ❑

3.10 Boolean properness

Theorem $\equiv$-truth: $\Delta(E \equiv F)$

Theorem ΔTrue

Theorem ΔFalse

Proof:

ΔFalse

$\equiv$"$\Delta\mathbb{B}$"

(False $\equiv$ True) $\equiv$ False

$\equiv$"$\equiv$-mirror, not False"

(False $\not\equiv$ True) $\equiv$ True

$\equiv$"two values"

True $\equiv$ True — $\equiv$-refl ❑

Theorem Δ-intro: $P \Rightarrow \Delta P$

Proof: $\Rightarrow$-defn and $\vee$-subst. ❑

Theorem $\Delta\Delta$: $\Delta\Delta P$

Proof:

$\Delta\Delta P$

$\equiv$"$\Delta\mathbb{B}$"

$(\Delta P \equiv \text{True}) \equiv \Delta P$

$\equiv$"$\Delta\mathbb{B}$"

$(((P \equiv \text{True}) \equiv P) \equiv \text{True}) \equiv ((P \equiv \text{True}) \equiv P)$ — $\equiv$-truth ❑

Theorem establishment: $(P \equiv \text{True}) \equiv \Delta P \wedge P$

Proof:

$\Delta P \wedge P$

$\equiv$"$\Delta\mathbb{B}$"

$((P \equiv \text{True}) \equiv P) \wedge P$

$\equiv$"$\wedge$-subst"

$((P \equiv \text{True}) \equiv P) \wedge (P \equiv \text{True})$

$\equiv$"$\wedge$-subst"

$((\text{True} \equiv \text{True}) \equiv \text{True}) \wedge (P \equiv \text{True})$

$\equiv$"$\equiv$-refl twice, $\wedge$-unit"

$P \equiv \text{True}$ ❑

Theorem establishment: $(P \equiv False) \equiv \Delta(\neg P) \wedge \neg P$

Proof:

$P \equiv False$

$\equiv$"False-defn"

$P \equiv \neg True$

$\equiv$"$\neg$-exchange"

$\neg P \equiv True$

$\equiv$"establishment above"

$\Delta(\neg P) \wedge \neg P$ ❑

Theorem $\Rightarrow$-defn: $P \Rightarrow Q \equiv \neg\Delta P \vee \neg P \vee Q$

Theorem included middle: $\neg\Delta P \vee \neg P \vee P$

Theorem excluded middle: $\Delta P \Rightarrow P \vee \neg P$

Proof:

$\Delta P \Rightarrow P \vee \neg P$

$\equiv$"$\Rightarrow$-defn"

$\neg\Delta P \vee \neg\Delta\Delta P \vee P \vee \neg P$

$\equiv$"$\Delta\Delta P$, elementary properties of $\vee$"

$\neg\Delta P \vee P \vee \neg P$ — included middle ❑

Theorem strong $\equiv$: $(E \equiv F) \vee (E \not\equiv F)$

Theorem: $P \vee \neg P \Rightarrow \Delta P$

Proof:

$P \vee \neg P \Rightarrow \Delta P$

$\equiv$"$\vee$-lub"

$(P \Rightarrow \Delta P) \wedge (\neg P \Rightarrow \Delta P)$

$\equiv$"$P \Rightarrow \Delta P$, $\wedge$-unit"

$\neg P \Rightarrow \Delta P$

$\equiv$"$\Rightarrow$-defn"

$(\neg P \not\equiv True) \vee \Delta P$

$\equiv$"$\equiv$-mirror"

$(P \not\equiv False) \vee \Delta P$

$\equiv$"$\vee$-subst"

$(P \not\equiv False) \vee \Delta False$

$\equiv$"ΔFalse, $\vee$-zero"

True ❑

Theorem $\Delta\neg$: $\Delta P \Rightarrow \Delta(\neg P)$

Proof:

ΔP

$\Rightarrow$"excluded middle"

$P \vee \neg P$

$\equiv$"$\vee$-symm, $\neg$-inv"

$\neg P \vee \neg\neg P$

$\Rightarrow$"$R \vee \neg R \Rightarrow \Delta R$ with $R := \neg P$"

$\Delta(\neg P)$ ❑

Theorem ≢-truth: $\Delta(E \not\equiv F)$

Theorem ≡-Falsity: $((E \equiv F) \equiv \text{False}) \equiv (E \not\equiv F)$

Proof:

$(E \equiv F) \equiv \text{False}$

≡"≡-mirror"

$(E \not\equiv F) \equiv \text{True}$

≡"$\Delta(E \not\equiv F)$"

$E \not\equiv F$ ❑

Theorem excluded middle: $\Delta P \Rightarrow (P \equiv \text{True}) \vee (P \equiv \text{False})$

Proof:

ΔP

$\Rightarrow$"excluded middle above, $\Delta P \Rightarrow \Delta P$, ⇒-combination"

$(P \vee \neg P) \wedge \Delta P$

$\Rightarrow$ "∧/∨"

$(\Delta P \wedge P) \vee (\Delta P \wedge \neg P)$

$\Rightarrow$"$\Delta P \Rightarrow \Delta(\neg P)$, ∧– and ∨-mono"

$(\Delta P \wedge P) \vee (\Delta(\neg P) \wedge \neg P)$

≡"establishment"

$(P \equiv \text{True}) \vee (P \equiv \text{False})$ ❑

3.11 One proper subterm and inference rules for case analysis

We have the following inference rule:

Case analysis $$\frac{Q[x:=\text{True}],\ Q[x:=\text{False}]}{\Delta P \Rightarrow Q[x:=P]}$$

Its validity follows straightforwardly from $\Delta P \Rightarrow (P \equiv \text{True}) \vee (P \equiv \text{False})$, ∨-lub, and ⇒-subst. A variation on this is as follows:

Case analysis $$\frac{\Delta P,\ P \Rightarrow Q,\ \neg P \Rightarrow Q}{Q}$$

Theorem ≡-unit: $(P \equiv \text{True}) \equiv P$ if ΔP

Proof:

$P \equiv \text{True}$

≡"establishment"

$\Delta P \wedge P$

≡"ΔP, ∧-unit"

P ❑

Theorem ≡-¬: $(P \equiv \text{False}) \equiv \neg P$ if ΔP

Theorem excluded middle: $P \vee \neg P$ if ΔP

Theorem (i) ∨/≡: $(P \vee (Q \equiv R)) \equiv (P \vee) \equiv P \vee R)$ if ΔP
(ii) ∧/≢: $(P \wedge (Q \not\equiv R)) \equiv (P \wedge Q \not\equiv P \wedge R)$ if ΔP

Proof:

(i) $P \vee (Q \equiv R)$
≡"ΔP, ⇒-defn"
$\neg P \Rightarrow (Q \equiv R)$
≡"⇒/≡"
$(\neg P \Rightarrow Q) \equiv (\neg P \Rightarrow R)$
≡"ΔP, ⇒-defn"
$(P \vee Q) \equiv (P \vee R)$

(ii) $(P \wedge (Q \not\equiv R))$
≡"de Morgan, ≢-defn, ¬-inv"
$\neg(\neg P \vee (Q \equiv R))$
≡"≡-mirror"
$\neg(\neg P \vee (\neg Q \equiv \neg R))$
≡"ΔP, (i)"
$\neg((\neg P \vee \neg Q) \equiv (\neg P \vee \neg R))$
≡"de Morgan"
$\neg(\neg(P \wedge Q) \equiv \neg(P \wedge R))$
≡"≡-mirror"
$\neg(P \wedge Q \equiv P \wedge R)$
≡"≢-defn"
$(P \wedge Q) \not\equiv (P \wedge R)$ ❑

Theorem* ∨/∧: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ if ΔP
Theorem* ∧/∨: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ if ΔP
Proof: Case analysis. ❑

Theorem absorption:
(i) $P \wedge (\neg P \vee Q) \equiv P \wedge Q$ if ΔP
(ii) $P \vee (\neg P \wedge Q) \equiv P \vee Q$ if ΔP
(iii) $P \wedge (P \Rightarrow Q) \equiv (P \wedge Q)$ if ΔP

Proof:

(i) $P \wedge (\neg P \vee Q)$
≡"∧ /∨ using ΔP"
$(P \wedge \neg P) \vee (P \wedge Q)$
≡"ΔP, excluded middle, de Morgan, ¬-inv, ∨–unit"
$P \wedge Q$

---

* The side-condition ΔP is redundant in 2- and 3-valued logics. See Section 5.

(ii) $P \vee (\neg P \wedge Q)$

$\equiv$"$\vee/\wedge$ using ΔP"

$(P \vee \neg P) \wedge (P \vee Q)$

$\equiv$"ΔP, excluded middle, $\wedge$–unit"

$P \vee Q$

(iii) $P \wedge (P \Rightarrow Q)$

$\equiv$"$\Rightarrow$-defn, ΔP"

$P \wedge (\neg P \vee Q)$

$\equiv$"ΔP, (i)"

$(P \wedge Q)$ ❑

Theorem: $(P \Rightarrow (Q \equiv R)) \equiv (P \wedge Q \equiv P \wedge R)$ if ΔP

Proof: Theorem $\Rightarrow/\equiv/\wedge$. ❑

Theorem Shannon: $Q[x:=P] \equiv (P \wedge Q[x:=True]) \vee (\neg P \wedge Q[x:=False])$ if ΔP

Proof: Case analysis on P. ❑

Theorem $P \not\equiv \neg P$ if ΔP

Proof:

$P \not\equiv \neg P$

$\equiv$"Shannon"

$(P \wedge (P \not\equiv \neg P)) \vee (\neg P \wedge (P \not\equiv \neg P))$

$\equiv$"ΔP"

$((P \equiv True) \wedge (P \not\equiv \neg P)) \vee ((P \equiv False) \wedge (P \not\equiv \neg P))$

$\equiv$"$\wedge$-subst, False-defn"

$((P \equiv True) \wedge (True \not\equiv False)) \vee ((P \equiv False) \wedge (False \not\equiv True))$

$\equiv$"two values, $\wedge$-unit"

$(P \equiv True) \vee (P \equiv False)$

$\Leftarrow$"excluded middle"

ΔP — given ❑

Theorem* consistency: $(P \wedge Q \equiv P) \equiv (P \vee Q \equiv Q)$ if ΔP or ΔQ

3.12 Proper subterms and derived inference rule bi-implication

Theorem $\equiv$**-**assoc: $((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$ if ΔP, ΔQ, ΔR

Proof:

$(P \equiv Q) \equiv R$

$\equiv$"ΔQ, Shannon"

$(Q \wedge ((P \equiv True) \equiv R)) \vee (\neg Q \wedge ((P \equiv False) \equiv R))$

$\equiv$"ΔP, and hence $\Delta(\neg P)$, $\equiv$-unit, $\equiv$-$\neg$"

$(Q \wedge (P \equiv R)) \vee (\neg Q \wedge (\neg P \equiv R))$

---

* The side-conditions ΔP and ΔQ are redundant in 2- and 3-valued logics. See Section 5.

≡"≡-mirror"

$(Q \wedge (P \equiv R)) \vee (\neg Q \wedge (P \equiv \neg R))$

≡" ΔR, ≡-unit, ≡-¬"

$(Q \wedge (P \equiv (\text{True} \equiv R))) \vee (\neg Q \wedge (P \equiv (\text{False} \equiv R)))$

≡"ΔQ, Shannon"

$P \equiv (Q \equiv R)$ ❑

Theorem ≢-assoc: $((P \not\equiv Q) \not\equiv R) \equiv (P \not\equiv (Q \not\equiv R))$ if ΔP, ΔQ, ΔR

Theorem ≡-≢-assoc (i) $((P \not\equiv Q) \equiv R) \equiv (P \not\equiv (Q \equiv R))$ if ΔP, ΔQ, ΔR

(ii) $((P \equiv Q) \not\equiv R) \equiv (P \equiv (Q \not\equiv R))$ if ΔP, ΔQ, ΔR

Theorem (i) ⇒-∧: $P \Rightarrow Q \equiv (P \wedge Q \equiv P)$ if ΔP, ΔQ

(ii) ⇒-∨: $P \Rightarrow Q \equiv (P \vee Q \equiv Q)$ if ΔP, ΔQ

Proof:

(ii) $(P \Rightarrow Q) \equiv (P \vee Q \equiv Q)$

≡"≡-assoc"

$(P \Rightarrow Q \equiv P \vee Q) \equiv Q$

≡"⇒-defn, ΔP"

$(\neg P \vee Q \equiv P \vee Q) \equiv Q$

≡"∨/≡, ΔQ"

$((\neg P \equiv P) \vee Q) \equiv Q$

≡"ΔP ⇒ (P ≢ ¬P)"

$\text{False} \vee Q \equiv Q$ — ∨-unit ❑

Theorem bi-implication: $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \equiv (P \equiv Q)$ if ΔP, ΔQ

Proof:

$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

≡"Δ, ⇒-∧"

$(P \wedge Q \equiv P) \wedge (Q \wedge P \equiv Q)$

≡"∧-symm"

$(P \wedge Q \equiv P) \wedge (P \wedge Q \equiv Q)$

≡"∧-subst"

$(P \wedge Q \equiv P) \wedge (P \equiv Q)$

≡"∧-subst"

$(P \wedge P \equiv P) \wedge (P \equiv Q)$

≡"∧-idem, ∧-unit"

$P \equiv Q$ ❑

Theorem contrapositive: (i) $P \Rightarrow \neg Q \equiv Q \Rightarrow \neg P$ if ΔP, ΔQ

(ii) $\neg P \Rightarrow Q \equiv \neg Q \Rightarrow P$ if ΔP, ΔQ

Proof:

(i) $(P \Rightarrow \neg Q) \equiv (Q \Rightarrow \neg P)$

≡"⇒-defn, ΔP, ΔQ"

$(\neg P \vee \neg Q) \equiv (\neg Q \vee \neg P)$ — ∨-symm ❑

Theorem ¬≡: $\neg(P \equiv Q) \equiv (\neg P \equiv Q)$ if $\Delta P, \Delta Q$

Proof

$\neg(P \equiv Q)$

≡"≡-falsity"

$(P \equiv Q) \equiv False$

≡"≡-assoc, ΔP, ΔQ"

$P \equiv (Q \equiv False)$

≡"establishment, ΔQ"

$P \equiv \neg Q$

≡"exchange and ≡-symm"

$\neg P \equiv Q$ ❑

Theorem contrapositive: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$ if $\Delta P, \Delta Q$

Theorem absorption: $P \wedge (P \equiv Q) \equiv P \wedge Q$ if $\Delta P, \Delta Q$

Proof:

$P \wedge (P \equiv Q)$

≡"bi-implication"

$P \wedge (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

≡"absorbtion"

$P \wedge Q \wedge (Q \Rightarrow P)$

≡" absorbtion"

$P \wedge Q \wedge P$

≡"∧-symm, ∧-idem"

$P \wedge Q$ ❑

Theorem ≡-𝔹: $(P\equiv Q) \equiv (P\vee\neg Q) \wedge (\neg P\vee Q)$ if $\Delta P, \Delta Q$

Proof: Rewrite the disjunctions as implications. ❑

Theorem ≡-𝔹: $(P\equiv Q) \equiv (P\wedge Q) \vee (\neg P\wedge\neg Q)$ if $\Delta P, \Delta Q$

Proof: Apply ∨/∧ to right-hand side . ❑

In classical 2-valued logic, equivalence can be shown via mutual implication. In the case of our many-valued logics, we require additionally that both sides be proper:

$$\text{bi-implication} \qquad \frac{P \Rightarrow Q,\ \ Q \Rightarrow P,\ \ \Delta P,\ \ \Delta Q}{P \equiv Q}$$

This is easily justified by the bi-implication theorem.

3.13 Further theorems on properness

Theorem Δ¬: $\Delta(\neg P) \equiv \Delta P$

Proof: ΔΔ and bi-implication. ❑

Theorem boolean properness: $\Delta P \equiv (P \equiv True) \vee (P \equiv False)$

Theorem $\Delta P \equiv \neg(P \Rightarrow \neg P) \vee \neg(\neg P \Rightarrow P))$

Proof:

$\neg(P \Rightarrow \neg P) \vee \neg(\neg P \Rightarrow P))$

≡"⇒-defn"

$\neg((P \not\equiv True) \vee \neg P) \vee \neg((\neg P \not\equiv True) \vee P)$

≡"de Morgan"

$((P \equiv True) \wedge P) \vee ((\neg P \equiv True) \wedge \neg P)$

≡"∧-subst"

$((P \equiv True) \wedge True) \vee ((\neg P \equiv True) \wedge True)$

≡"∧-unit"

$(P \equiv True) \vee (\neg P \equiv True)$

≡"exchange"

$(P \equiv True) \vee (P \equiv False)$

≡"preceding theorem"

ΔP ❑

4. Predicate E⁻

4.1 Derived inference rules for quantifications

Proofs in predicate logic use the following set of derived inference rules:

∀-Leibniz 1 $$\frac{t \equiv u}{(\forall x{:}T|R[y{:=}t] \bullet P[y{:=}t]) \equiv (\forall x{:}T|R[y{:=}u] \bullet P[y{:=}u])}$$

∀-Leibniz 2 $$\frac{R \Rightarrow (t \equiv u)}{(\forall x{:}T|R \bullet P[y{:=}t]) \equiv (\forall x{:}T|R \bullet P[y{:=}u])}$$

We also have the rules ∃-Leibniz 1 and ∃-Leibniz 2 by replacing ∀ with ∃ in the above, respectively. In the preceding rules, it is not excluded that x and y are identical symbols.

∀-mono 1 $$\frac{R \wedge P \Rightarrow Q}{(\forall x{:}T|R \bullet P) \Rightarrow (\forall x{:}T|R \bullet Q)}$$

∀-mono 2 $$\frac{R \Rightarrow S}{(\forall x{:}T|S \bullet P) \Rightarrow (\forall x{:}T|R \bullet P)}$$

We also have the rules ∃-mono 1 and ∃-mono 2 by replacing ∀ with ∃ in the above, respectively.

Instantiation $$\frac{(\forall x{:}T \bullet P)}{P}$$

We show that these inference rules can be derived from the axioms, modus ponens, and generalisation.

4.1.1 ∀-Leibniz

$$\frac{t \equiv u}{(\forall x{:}T|R[y{:=}t] \bullet P[y{:=}t]) \equiv (\forall x{:}T|R[y{:=}u] \bullet P[y{:=}u])}$$

where x and y may stand for the same variable.

Proof:

(i)	$t \equiv u$	— hypothesis
(ii)	$(t \equiv u) \Rightarrow (R[y{:=}t] \Rightarrow P[y{:=}t] \equiv R[y{:=}u] \Rightarrow P[y{:=}u])$	— axiom Leibniz
(iii)	$R[y{:=}t] \Rightarrow P[y{:=}t] \equiv R[y{:=}u] \Rightarrow P[y{:=}u]$	— (i), (ii), MP
(iv)	$(\forall x{:}T \bullet R[y{:=}t] \Rightarrow P[y{:=}t] \equiv R[y{:=}u] \Rightarrow P[y{:=}u])$	— (iii), generalisation
(v)	$(iv) \Rightarrow (\forall x{:}T \bullet R[y{:=}t] \Rightarrow P[y{:=}t]) \equiv (\forall x{:}T \bullet R[y{:=}u] \Rightarrow P[y{:=}u])$	— axiom $\forall/\equiv$
(vi)	$(\forall x{:}T \bullet R[y{:=}t] \Rightarrow P[y{:=}t]) \equiv (\forall x{:}T \bullet R[y{:=}u] \Rightarrow P[y{:=}u])$	— (iv), (v), MP
(vii)	$(\forall x{:}T|R[y{:=}t] \bullet P[y{:=}t]) \equiv (\forall x{:}T|R[y{:=}u] \bullet P[y{:=}u])$	— trading and transitivity

Note that the proof does not require that x and y be distinct variables. Obvious variations are

$$\frac{t \equiv u}{(\forall x{:}T \bullet P[y{:=}t]) \equiv (\forall x{:}T \bullet P[y{:=}u])}$$

$$\frac{R \Rightarrow (t \equiv u)}{(\forall x{:}T|R \bullet P[y{:=}t]) \equiv (\forall x{:}T|R \bullet P[y{:=}u])}$$

where again x and y need not be distinct variables.

4.1.2 ∃-Leibniz

There are analogous inference rules for ∃:

$$\frac{t \equiv u}{(\exists x{:}T|R[y{:=}t] \bullet P[y{:=}t]) \equiv (\exists x{:}T|R[y{:=}t] \bullet Q[y{:=}u])}$$

$$\frac{t \equiv u}{(\exists x{:}T \bullet P[y{:=}t]) \equiv (\exists x{:}T \bullet Q[y{:=}u])}$$

$$\frac{R \Rightarrow (t \equiv u)}{(\exists x{:}T|R \bullet P[y{:=}t]) \equiv (\exists x{:}T|R \bullet Q[y{:=}u])}$$

The proofs are as above, except that they rely on

$$(\forall x{:}T|R \bullet P \equiv Q) \Rightarrow ((\exists x{:}T|R \bullet P) \equiv (\exists x{:}T|R \bullet Q)) \quad (*)$$

is used instead of $\forall/\equiv$. (*) is indeed a theorem and can be proved without recourse to these inference rules.

4.1.3 ∀-monotonicity

$$\frac{P \Rightarrow Q}{(\forall x{:}T \bullet P) \Rightarrow (\forall x{:}T \bullet Q)}$$

$$\frac{R \wedge P \Rightarrow Q}{(\forall x{:}T|R \bullet P) \Rightarrow (\forall x{:}T|R \bullet Q)}$$

$$\frac{R \Rightarrow S}{(\forall x{:}T|S \bullet P) \Rightarrow (\forall x{:}T|R \bullet P)}$$

Proof of first rule:

(i)	$P \Rightarrow Q$	— hypothesis
(ii)	$(\forall x{:}T \bullet P \Rightarrow Q)$	— (i), generalisation
(iii)	$(\forall x{:}T \bullet P \Rightarrow Q) \Rightarrow ((\forall x{:}T \bullet P) \Rightarrow (\forall x{:}T \bullet Q))$	— theorem $\forall/\Rightarrow$
(iv)	$(\forall x{:}T \bullet P) \Rightarrow (\forall x{:}T \bullet Q)$	— (ii), (iii), MP

Theorem $\forall/\Rightarrow$ is

$$(\forall x{:}T|R \bullet P \Rightarrow Q) \Rightarrow ((\forall x{:}T|R \bullet P) \Rightarrow (\forall x{:}T|R \bullet Q))$$

and can be proved without recourse to these inference rules; the proof is presented below.

4.1.4 ∃-monotonicity

The analogous inference rules for ∃ are

$$\frac{P \Rightarrow Q}{(\exists x{:}T \bullet P) \Rightarrow (\exists x{:}T \bullet Q)}$$

$$\frac{R \wedge P \Rightarrow Q}{(\exists x{:}T|R \bullet P) \Rightarrow (\exists x{:}T|R \bullet Q)}$$

$$\frac{R \Rightarrow S}{(\exists x{:}T|R \bullet P) \Rightarrow (\exists x{:}T|S \bullet P)}$$

The proofs are as for ∀-mono except that they rely on the theorem

$$(\forall x{:}T \bullet P \Rightarrow Q) \Rightarrow ((\exists x{:}T \bullet P) \Rightarrow (\exists x{:}T \bullet Q))$$

and similar which can be proved without recourse to the inference rules.

4.1.5 Instantiation

$$\frac{(\forall x{:}T \bullet P)}{P}$$

Its proof relies on $(\forall x{:}T \bullet P) \Rightarrow P$, which can be proved without recourse to instantiation, and MP.

4.1.6 Reasoning with quantifications

With the quantifiers, there are some additional ways of justifying steps in proof presentations. A justification of $(\forall x{:}T|R \bullet P) \equiv (\forall x{:}T|R \bullet Q)$ may simply be a justification of $P \equiv Q$ or $R \Rightarrow (P \equiv Q)$ — the legitimacy of this follows immediately from the derived inference rule ∀-Leibniz. A similar remark holds for existential quantification. Furthermore, in both of

$(\forall x{:}T|R \bullet P)$	$(\exists x{:}T|R \bullet P)$
⇒"justification 1"	⇒"justification 1"
$(\forall x{:}T|R \bullet Q)$	$(\exists x{:}T|R \bullet Q)$

"justification 1" may be a justification of $P \Rightarrow Q$ or $R \wedge P \Rightarrow Q$. The legitimacy of these follows easily from ∀- and ∃-mono, respectively.

All of the above holds good when the quantifications have no ranges.

4.2 Utility theorems for predicate logic

Theorem ∃-truth: $((\exists x{:}T \bullet P) \equiv \text{True}) \equiv (\exists x{:}T \bullet P \equiv \text{True})$

Proof:

$(\exists x{:}T \bullet P) \equiv \text{True}$

≡"≢-defn, ¬-inv"

$\neg((\exists x{:}T \bullet P) \not\equiv \text{True})$

≡"∨-unit"

$\neg((\exists x{:}T \bullet P) \not\equiv \text{True}) \vee \text{False})$

≡"⇒-defn"

$\neg((\exists x{:}T \bullet P) \Rightarrow \text{False})$

≡"∃-lub"

$\neg(\forall x{:}T \bullet P \Rightarrow \text{False})$

≡"⇒-defn, ∨-unit"

$\neg(\forall x{:}T \bullet P \not\equiv \text{True})$

≡"de Morgan"

$(\exists x{:}T \bullet P \equiv \text{True})$ ❑

Theorem universal range: $(\forall x{:}T \bullet P) \equiv (\forall x{:}T|\text{True} \bullet P)$

Theorem universal range: $(\exists x{:}T \bullet P) \equiv (\exists x{:}T|\text{True} \bullet P)$

Theorem de Morgan: (i) $\neg(\exists x{:}T|R \bullet P) \equiv (\forall x{:}T|R \bullet \neg P)$
(ii) $\neg(\forall x{:}T|R \bullet P) \equiv (\exists x{:}T|R \bullet \neg P)$

Proof of (i):

$\neg(\exists x{:}T|R \bullet P)$
≡"∃|-defn"
$\neg\neg(\forall x{:}T|R \bullet \neg P)$
≡"¬-inv"
$(\forall x{:}T|R \bullet \neg P)$ ❑

Theorem instantiation: $(\forall x{:}T \bullet P) \Rightarrow P$

Proof: instantiation and Δx. ❑

Theorem instantiation: $P \Rightarrow (\exists x{:}T \bullet P)$

Proof (using True-elim)

$P \Rightarrow (\exists x{:}T \bullet P) \equiv True$
≡"⇒-truth"
$(P \equiv True) \Rightarrow ((\exists x{:}T \bullet P) \equiv True)$
≡"contrapositive, Δ≡"
$\neg((\exists x{:}T \bullet P) \equiv True) \Rightarrow (P \not\equiv True)$
≡"∃-truth"
$\neg(\exists x{:}T \bullet P \equiv True) \Rightarrow (P \not\equiv True)$
≡"de Morgan"
$(\forall x{:}T \bullet P \not\equiv True) \Rightarrow (P \not\equiv True)$ — preceding instantiation ❑

4.3 Monotonicity of quantifications

Theorem ∀/≡: $(\forall x{:}T|R \bullet P \equiv Q) \Rightarrow ((\forall x{:}T|R \bullet P) \equiv (\forall x{:}T|R \bullet Q))$

Proof:

$(\forall x{:}T|R \bullet P \equiv Q)$
≡"trading"
$(\forall x{:}T \bullet R \Rightarrow (P \equiv Q))$
≡"⇒/≡"
$(\forall x{:}T \bullet R \Rightarrow P \equiv R \Rightarrow Q)$
⇒"axiom ∀/≡"
$(\forall x{:}T \bullet R \Rightarrow P) \equiv (\forall x{:}T \bullet R \Rightarrow Q)$
≡"trading"
$(\forall x{:}T|R \bullet P) \equiv (\forall x{:}T|R \bullet Q)$ ❑

Theorem ∃/≡: $(\forall x{:}T|R \bullet P \equiv Q) \Rightarrow ((\exists x{:}T|R \bullet P) \equiv (\exists x{:}T|R \bullet Q))$

Proof:

$(\exists x{:}T|R \bullet P) \equiv (\exists x{:}T|R \bullet Q)$
≡"∃|-defn"
$\neg(\forall x{:}T|R \bullet \neg P) \equiv \neg(\forall x{:}T|R \bullet \neg Q)$
≡"≡-mirror"
$(\forall x{:}T|R \bullet \neg P) \equiv (\forall x{:}T|R \bullet \neg Q)$
⇐"∀/≡'

(∀x:T|R • ¬P ≡ ¬Q)
≡"≡-mirror"
(∀x:T|R • P ≡ Q) ❑

Theorem ∀/⇒: (∀x:T|R • P ⇒ Q) ⇒ ((∀x:T|R • P) ⇒ (∀x:T|R • Q))
Proof:
(∀x:T|R • P ⇒ Q) ⇒ ((∀x:T|R • P) ⇒ (∀x:T|R • Q))
≡"trading"
(∀x:T • R ⇒ (P ⇒ Q)) ⇒ ((∀x:T|R • P) ⇒ (∀x:T|R • Q))
≡"⇒/⇒"
(∀x:T • (R ⇒ P) ⇒ (R ⇒ Q)) ⇒ ((∀x:T|R • P) ⇒ (∀x:T|R • Q))
≡"trading twice"
(∀x:T • (R ⇒ P) ⇒ (R ⇒ Q)) ⇒
((∀x:T • R ⇒ P) ⇒ (∀x:T • R ⇒ Q))

Hence we need only prove the case where there is no range.

(∀x:T • P ⇒ Q) ⇒ ((∀x:T • P) ⇒ (∀x:T • Q))
≡"shunting"
(∀x:T • P ⇒ Q) ∧ (∀x:T • P) ⇒ (∀x:T • Q)
≡"∀/∧"
(∀x:T • (P ⇒ Q) ∧ P) ⇒ (∀x:T • Q)
≡"True-⇒"
(∀x:T • ((P ⇒ Q) ∧ P) ≡ True) ⇒ (∀x:T • Q)
≡"∀-truth, ∧-truth, ⇒-truth"
(∀x:T • ((P ≡ True) ⇒ (Q ≡ True)) ∧ (P ≡ True)) ⇒ (∀x:T • Q)
≡"absorption, Δ(P ≡ True)"
(∀x:T • (P ≡ True) ∧ (Q ≡ True)) ⇒ (∀x:T • Q)
≡"∧-truth, ∀-truth"
((∀x:T • P ∧ Q) ≡ True) ⇒ (∀x:T • Q)
≡"True-⇒"
(∀x:T • P ∧ Q) ⇒ (∀x:T • Q)
≡"∀/∧"
(∀x:T • P) ∧ (∀x:T • Q) ⇒ (∀x:T • Q) — weakening ❑

Theorem ∃/⇒: (∀x:T|R • P ⇒ Q) ⇒ ((∃x:T|R • P) ⇒ (∃x:T|R • Q))
Proof (for case of no range):
(∃x:T • P) ⇒ (∃x:T • Q)
≡"∃-lub"
(∀x:T • P ⇒ (∃x:T • Q))
⇐ "Q ⇒ (∃x:T • Q)"
(∀x:T • P ⇒ Q) ❑

Theorem ∀ range anti-mono: (∀x:T • R ⇒ S) ⇒ ((∀x:T|S • P) ⇒ (∀x:T|R • P))
Proof:
(∀x:T|S • P) ⇒ (∀x:T|R • P)

$\equiv$“trading twice”

$(\forall x{:}T \bullet S \Rightarrow P) \Rightarrow (\forall x{:}T \bullet R \Rightarrow P))$

$\Leftarrow$“$\forall/\Rightarrow$”

$(\forall x{:}T \bullet (S \Rightarrow P) \Rightarrow (R \Rightarrow P))$

$\Leftarrow$“$\Rightarrow$-left-anti-mono”

$(\forall x{:}T \bullet R \Rightarrow S)$ ❑

Theorem $\exists$ range mono: $(\forall x{:}T \bullet S \Rightarrow R) \;\Rightarrow\; ((\exists x{:}T|S \bullet P) \Rightarrow (\exists x{:}T|R \bullet P))$

4.4 Range manipulation

Theorem trading: $(\exists x{:}T|R \bullet P) \;\equiv\; (\exists x{:}T \bullet (R \equiv True) \wedge P))$

Proof:

$(\exists x{:}T|R \bullet P)$

$\equiv$“$\exists|$-defn”

$\neg(\forall x{:}T|R \bullet \neg P))$

$\equiv$“trading for $\forall$”

$\neg(\forall x{:}T \bullet R \Rightarrow \neg P))$

$\equiv$“$\exists$-defn”

$(\exists x{:}T \bullet \neg(R \Rightarrow \neg P))$

$\equiv$“$\Rightarrow$-defn”

$(\exists x{:}T \bullet \neg(R \not\equiv True) \vee \neg P)$

$\equiv$“de Morgan”

$(\exists x{:}T \bullet (R \equiv True) \wedge P)$ ❑

Theorem range split: $(\forall x{:}T|R\vee S \bullet P) \;\equiv\; (\forall x{:}T|R \bullet P) \wedge (\forall x{:}T|S \bullet P)$

Proof:

$(\forall x{:}T|R \bullet P) \wedge (\forall x{:}T|S \bullet P)$

$\equiv$“trading”

$(\forall x{:}T \bullet R \Rightarrow P) \wedge (\forall x{:}T \bullet S \Rightarrow P)$

$\equiv$“$\forall/\wedge$”

$(\forall x{:}T \bullet (R \Rightarrow P) \wedge (S \Rightarrow P))$

$\equiv$“$\vee$-lub”

$(\forall x{:}T \bullet R\vee S \Rightarrow P)$

$\equiv$“trading”

$(\forall x{:}T|R\vee S \bullet P)$ ❑

Theorem range split: $(\exists x{:}T|R\vee S \bullet P) \;\equiv\; (\exists x{:}T|R \bullet P) \vee (\exists x{:}T|S \bullet P)$

Theorem partial trading: $(\forall x{:}T|R\wedge S \bullet P) \;\equiv\; (\forall x{:}T|R \bullet S \Rightarrow P)$

Proof:

$(\forall x{:}T|R\wedge S \bullet P)$

$\equiv$“trading”

$(\forall x{:}T \bullet R \wedge S \Rightarrow P)$

$\equiv$“shunting’

$(\forall x{:}T \bullet R \Rightarrow (S \Rightarrow P))$

≡“trading”

$(\forall x{:}T|R \bullet S \Rightarrow P)$ ❑

Theorem partial trading: $(\exists x{:}T|R{\wedge}S \bullet P) \equiv (\exists x{:}T|R \bullet (S{\equiv}True) \wedge P)$

4.5 Truth and falsity in quantifications

Theorem ∀-truth: $((\forall x{:}T|R \bullet P) \equiv True) \equiv (\forall x{:}T|R \bullet P \equiv True)$

Theorem ∃-truth: $((\exists x{:}T|R \bullet P) \equiv True) \equiv (\exists x{:}T|R \bullet P \equiv True)$

Theorem ∀-falsity: $((\forall x{:}T|R \bullet P) \equiv False) \equiv (\exists x{:}T|R \bullet P \equiv False)$

Theorem ∃-falsity: $((\exists x{:}T|R \bullet P) \equiv False) \equiv (\forall x{:}T|R \bullet P \equiv False)$

Theorem ∀-non-truth: $((\forall x{:}T|R \bullet P) \not\equiv True) \equiv (\exists x{:}T|R \bullet P \not\equiv True)$

Theorem ∃-non-truth: $((\exists x{:}T|R \bullet P) \not\equiv True) \equiv (\forall x{:}T|R \bullet P \not\equiv True)$

4.6 Constant predicate

Theorem ∀-constant: $(\forall x{:}T|R \bullet P) \equiv P \vee (\forall x{:}T \bullet R{\not\equiv}True)$ if x does not occur free in P.

Proof:

$(\forall x{:}T|R \bullet P)$

≡“trading”

$(\forall x{:}T \bullet R \Rightarrow P)$

≡“∃-lub”

$(\exists x{:}T \bullet R) \Rightarrow P$

≡“⇒-defn, ∃-non-truth”

$P \vee (\forall x{:}T \bullet R{\not\equiv}True)$ ❑

Theorem ∃-constant: $(\exists x{:}T|R \bullet P) \equiv P \wedge (\exists x{:}T \bullet R{\equiv}True)$ if x does not occur free in P.

Proof:

$(\exists x{:}T|R \bullet P)$

≡“∃|-defn”

$\neg(\forall x{:}T|R \bullet \neg P)$

≡“∀-constant”

$\neg(\neg P \vee (\forall x{:}T \bullet R{\not\equiv}True))$

≡“de Morgan”

$P \wedge (\exists x{:}T \bullet R{\equiv}True)$ ❑

Theorem ∀-idem: $(\forall x{:}T \bullet True)$

Proof: True and generalisation. ❑

Theorem habitation: $(\exists x{:}T \bullet True)$

Theorem habitation: $\neg(\forall x{:}T \bullet False)$

Theorem ∀-idem: $(\forall x{:}T \bullet P) \equiv P$ if x not free in P.

Theorem ∃-idem: $(\exists x{:}T \bullet P) \equiv P$ if x not free in P.

Theorem ∀-idem: $(\forall x{:}T|R \bullet \text{True})$

Theorem ∃-idem: $\neg(\exists x{:}T|R \bullet \text{False})$

Theorem habitation: $(\forall x{:}T|R \bullet \text{False}) \equiv (\forall x{:}T \bullet R \not\equiv \text{True})$

Theorem habitation: $(\exists x{:}T|R \bullet \text{True}) \equiv (\exists x{:}T \bullet R \equiv \text{True})$

Theorem empty range: $(\forall x{:}T|\text{False} \bullet P)$

Theorem empty range: $\neg(\exists x{:}T|\text{False} \bullet P)$

4.7 Distribution of quantifications

Theorem ∀/∧: $(\forall x{:}T|R \bullet P \wedge Q) \equiv (\forall x{:}T|R \bullet P) \wedge (\forall x{:}T|R \bullet Q)$

Theorem ∃/∨: $(\exists x{:}T|R \bullet P \vee Q) \equiv (\exists x{:}T|R \bullet P) \vee (\exists x{:}T|R \bullet Q)$

Proof:

$(\exists x{:}T|R \bullet P) \vee (\exists x{:}T|R \bullet Q)$

≡“de Morgan”

$\neg(\neg(\exists x{:}T|R \bullet P) \wedge \neg(\exists x{:}T|R \bullet Q))$

≡“de Morgan”

$\neg((\forall x{:}T|R \bullet \neg P) \wedge (\forall x{:}T|R \bullet \neg Q))$

≡“∀/∧”

$\neg(\forall x{:}T|R \bullet \neg P \wedge \neg Q)$

≡“de Morgan”

$(\exists x{:}T|R \bullet P \vee Q)$ ❑

Theorem ⇒/∀: $P \Rightarrow (\forall x{:}T|R \bullet Q) \equiv (\forall x{:}T|R \bullet P \Rightarrow Q)$ if x not free in P.

Proof:

$P \Rightarrow (\forall x{:}T|R \bullet Q)$

≡“trading”

$P \Rightarrow (\forall x{:}T \bullet R \Rightarrow Q)$

≡“⇒/∀”

$(\forall x{:}T \bullet P \Rightarrow (R \Rightarrow Q))$

≡“shunting, ∧-symm, shunting”

$(\forall x{:}T \bullet R \Rightarrow (P \Rightarrow Q))$

≡“trading”

$(\forall x{:}T|R \bullet P \Rightarrow Q)$ ❑

Theorem ∃-lub: $(\exists x{:}T|R \bullet P) \Rightarrow Q \equiv (\forall x{:}T|R \bullet P \Rightarrow Q)$ if x not free in Q.

Theorem* $\vee/\forall$: $P \vee (\forall x{:}T|R \bullet Q) \Leftrightarrow (\forall x{:}T|R \bullet P \vee Q)$ provided x does not occur free in P.
Proof (for the case of no range):

($\Rightarrow$) $P \vee (\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet P \vee Q)$
≡"∨-lub"
$(P \Rightarrow (\forall x{:}T \bullet P \vee Q)) \wedge ((\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet P \vee Q))$
≡"⇒/∀"
$(\forall x{:}T \bullet P \Rightarrow (P \vee Q)) \wedge ((\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet P \vee Q))$
≡"weakening, ∀-idem, ∧-unit"
$(\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet P \vee Q)$
⇐"∀/⇒"
$(\forall x{:}T \bullet Q \Rightarrow P \vee Q)$
≡"weakening, ∀-idem"
True

($\Leftarrow$) $((\forall x{:}T \bullet P \vee Q) \Rightarrow P \vee (\forall x{:}T \bullet Q)) \equiv \text{True}$ (using True-elim)
≡"⇒-truth"
$((\forall x{:}T \bullet P \vee Q) \equiv \text{True}) \Rightarrow ((P \vee (\forall x{:}T \bullet Q)) \equiv \text{True})$
≡"∀-truth (twice), ∨-truth "
$(\forall x{:}T \bullet (P \equiv \text{True}) \vee (Q \equiv \text{True})) \Rightarrow (P \equiv \text{True}) \vee (\forall x{:}T \bullet Q \equiv \text{True})$
≡"⇒-defn, (P ≡ True) ≡ ((P ≢ True) ≢ True)"
$(\forall x{:}T \bullet (P \equiv \text{True}) \vee (Q \equiv \text{True})) \Rightarrow ((P \not\equiv \text{True}) \Rightarrow (\forall x{:}T \bullet Q \equiv \text{True}))$
≡"⇒/∀"
$(\forall x{:}T \bullet (P \equiv \text{True}) \vee (Q \equiv \text{True})) \Rightarrow (\forall x{:}T \bullet (P \not\equiv \text{True}) \Rightarrow (Q \equiv \text{True}))$
≡"⇒-defn, (P ≡ True) ≡ ((P ≢ True) ≢ True)"
$(\forall x{:}T \bullet (P \equiv \text{True}) \vee (Q \equiv \text{True})) \Rightarrow (\forall x{:}T \bullet (P \equiv \text{True}) \vee (Q \equiv \text{True}))$
≡"⇒-reflex"
True ❑

Theorem* $\wedge/\exists$: $P \wedge (\exists x{:}T|R \bullet Q) \Leftrightarrow (\exists x{:}T|R \bullet P \wedge Q)$ provided x does not occur free in P.
Proof (for the case of no range):

($\Rightarrow$) $P \wedge (\exists x{:}T \bullet Q) \Rightarrow (\exists x{:}T \bullet P \wedge Q)$
≡"⇒-defn, ∃-non-truth, ∧-truth"
$(P \not\equiv \text{True}) \vee (\forall x{:}T \bullet Q \not\equiv \text{True}) \vee (\exists x{:}T \bullet P \wedge Q)$
≡"⇒-defn"
$(P \Rightarrow (\forall x{:}T \bullet Q \not\equiv \text{True})) \vee (\exists x{:}T \bullet P \wedge Q)$
≡"⇒/∀"
$(\forall x{:}T \bullet P \Rightarrow (Q \not\equiv \text{True})) \vee (\exists x{:}T \bullet P \wedge Q)$
≡"⇒-defn"
$(\forall x{:}T \bullet (P \not\equiv \text{True}) \vee (Q \not\equiv \text{True})) \vee (\exists x{:}T \bullet P \wedge Q)$
≡"∧-¬truth, ∃-non-truth,"
$((\exists x{:}T \bullet P \wedge Q) \not\equiv \text{True}) \vee (\exists x{:}T \bullet P \wedge Q)$
≡"⇒-defn, ⇒-reflex"
True

* The bi-implication can be replaced by equivalence in 2- and 3-valued logics. See Section 5.

(⇐) $(\exists x{:}T \bullet P \wedge Q) \Rightarrow P \wedge (\exists x{:}T \bullet Q)$

≡"⇒/∧"

$((\exists x{:}T \bullet P \wedge Q) \Rightarrow P) \wedge ((\exists x{:}T \bullet P \wedge Q) \Rightarrow (\exists x{:}T \bullet Q))$

≡"∃-lub"

$(\forall x{:}T \bullet P \wedge Q \Rightarrow P) \wedge ((\exists x{:}T \bullet P \wedge Q) \Rightarrow (\exists x{:}T \bullet Q))$

≡"weakening, ∀-idem, ∧-unit"

$(\exists x{:}T \bullet P \wedge Q) \Rightarrow (\exists x{:}T \bullet Q)$

⇐"∃/⇒"

$(\forall x{:}T \bullet P \wedge Q \Rightarrow Q)$

≡"weakening, ∀-idem"

True ❑

Theorem ∀/∨: $(\forall x{:}T|R \bullet P) \vee (\forall x{:}T|R \bullet Q) \Rightarrow (\forall x{:}T|R \bullet P \vee Q)$

Proof: ∀-mono twice. ❑

Theorem ∃/∧: $(\exists x{:}T|R \bullet P \wedge Q) \Rightarrow (\exists x{:}T|R \bullet P) \wedge (\exists x{:}T|R \bullet Q)$

Proof:

$(\exists x{:}T|R \bullet P \wedge Q) \Rightarrow (\exists x{:}T|R \bullet P) \wedge (\exists x{:}T|R \bullet Q)$

≡"⇒/∧"

$((\exists x{:}T|R \bullet P \wedge Q) \Rightarrow (\exists x{:}T|R \bullet P)) \wedge ((\exists x{:}T|R \bullet P \wedge Q) \Rightarrow (\exists x{:}T|R \bullet Q))$

⇐"∃/⇒"

$(\forall x{:}T|R \bullet P \wedge Q \Rightarrow P) \wedge (\forall x{:}T|R \bullet P \wedge Q \Rightarrow Q)$

≡"weakening"

$(\forall x{:}T|R \bullet \mathrm{True}) \wedge (\forall x{:}T|R \bullet \mathrm{True})$

≡"∀-idem"

True ❑

Theorem ∧/∀: $(\forall x{:}T|R \bullet P \wedge Q) \equiv P \wedge (\forall x{:}T|R \bullet Q)$ if x not free in P and $(\exists x{:}T \bullet R \equiv \mathrm{True})$

Proof:

$(\forall x{:}T|R \bullet P \wedge Q)$

≡"∀/∧"

$(\forall x{:}T|R \bullet P) \wedge (\forall x{:}T|R \bullet Q)$

≡"∀-constant"

$(P \vee (\forall x{:}T \bullet R \not\equiv \mathrm{True})) \wedge (\forall x{:}T|R \bullet Q)$

≡"$(\exists x{:}T \bullet R \equiv \mathrm{True})$, i.e. $\neg(\forall x{:}T \bullet R \not\equiv \mathrm{True})$"

$P \wedge (\forall x{:}T|R \bullet Q)$ ❑

Theorem ∨/∃: $(\exists x{:}T|R \bullet P \vee Q) \equiv P \vee (\exists x{:}T|R \bullet Q)$ if x not free in P and $(\exists x{:}T \bullet R \equiv \mathrm{True})$

Theorem ⇒/∃: $(\exists x{:}T|R \bullet P \Rightarrow Q) \equiv P \Rightarrow (\exists x{:}T|R \bullet Q)$ if x not free in P and $(\exists x{:}T \bullet R \equiv \mathrm{True})$

Proof:

$(\exists x{:}T|R \bullet P \Rightarrow Q)$

≡"⇒-defn"

$(\exists x{:}T|R \bullet (P \not\equiv \mathrm{True}) \vee Q)$

≡"∨/∃"

(P≢True) ∨ (∃x:T|R • Q)
≡"⇒-defn"
P ⇒ (∃x:T|R • Q) ❑

Theorem ∨/∀: P ∨ (∀x:T|R • Q) ≡ (∀x:T|R • P ∨ Q) if ΔP and x does not occur free in P.
Proof: By case analysis on P. ❑

Theorem ∧/∃: P ∧ (∃x:T|R • Q) ≡ (∃x:T|R • P ∧ Q) if ΔP and x does not occur free in P.

4.8 The deduction theorem for the predicate calculus

The deduction theorem, as used in classical predicate logic, allows us to conclude U, V, ..., ⊢ W⇒P from U, V, ..., W ⊢ P, provided that in the proof of U, V, ..., W ⊢ P generalisation is not used over variables which occur free in W. The deduction theorem continues to hold. The standard proof relies on the theorems mentioned in connection with the deduction theorem for propositional calculus, and in addition (∀x:T • P ⇒ Q) ⇒ (P ⇒ (∀x:T • Q)) for P not containing x. This holds by axiom ⇒/∀ and theorem ⇒-reflex.

4.9 Instantiation

Theorem instantiation: P[x:=t] ∧ Δt ⇒ (∃x:T • P)
Proof (using True-elim):
(P[x:=t] ∧ Δt ⇒ (∃x:T • P)) ≡ True
≡"⇒-truth, ⇒-True"
((P[x:=t] ∧ Δt) ≡ True) ⇒ ((∃x:T • P) ≡ True)
≡"contrapositive, Δ≡"
¬((∃x:T • P) ≡ True) ⇒ ¬((P[x:=t] ∧ Δt) ≡ True)
≡"∃-truth, ∧-truth"
¬(∃x:T • P ≡ True) ⇒ ¬((P[x:=t] ≡ True) ∧ (Δt ≡ True))
≡"∃-defn"
(∀x:T • P ≢ True) ⇒ (P[x:=t] ≢ True) ∨ (Δt ≢ True)
≡"⇒-defn"
(∀x:T • P ≢ True) ⇒ (Δt ⇒ (P[x:=t] ≢ True))
≡"shunting, substitution property"
(∀x:T • P ≢ True) ∧ Δt ⇒ (P ≢ True)[x:=t] — instantiation ❑

Theorem instantiation: (∀x:T|R • P) ⇒ (R ⇒ P)

Theorem instantiation: R ∧ P ⇒ (∃x:T|R • P)

4.10 Dummy manipulation

Theorem renaming: (∃x:T • P) ≡ (∃y:T • P[x:=y]) y fresh.

Theorem interchange: (∀x:T|R • (∀y:U|S • P)) ≡ (∀y:U|S • (∀x:T|R • P)) if x not in S and y not in R.

Theorem interchange: ((∃x:T|R • (∃y:U|S • P)) ≡ ((∃y:U|S • (∃x:T|R • P)) if x not in S and y not in R.

Theorem ∃/∀: (∃x:T|R • (∀y:U|S • P)) ⇒ (∀y:U|S • (∃x:T|R • P)) if x not in S and y not in R

Proof:

$(\exists x{:}T|R \bullet (\forall y{:}U|S \bullet P)) \;\Rightarrow\; (\forall y{:}U|S \bullet (\exists x{:}T|R \bullet P))$

≡"⇒/∀, y not free in $(\exists x{:}T|R \bullet (\forall y{:}U|S \bullet P))$"

$(\forall y{:}U|S \bullet (\exists x{:}T|R \bullet (\forall y{:}U|S \bullet P)) \;\Rightarrow\; (\exists x{:}T|R \bullet P))$

⇐"∃/⇒, ∀-mono"

$(\forall y{:}U|S \bullet (\forall x{:}T|R \bullet (\forall y{:}U|S \bullet P) \Rightarrow P))$

⇐"instantiation"

$(\forall y{:}U|S \bullet (\forall x{:}T|R \bullet (S \Rightarrow P) \Rightarrow P))$

≡"trading, ⇒/∀, x not free in S"

$(\forall y{:}U \bullet (\forall x{:}T|R \bullet (S \;\Rightarrow ((S \Rightarrow P) \Rightarrow P))))$

≡"shunting, modus ponens"

$(\forall y{:}U \bullet (\forall x{:}T \bullet True))$ — True generalised twice ❑

Theorem ∀∧∃: $(\forall x{:}T|R \bullet P) \wedge (\exists x{:}T|R \bullet Q) \Rightarrow (\exists x{:}T|R \bullet P \wedge Q)$

Proof:

$(\forall x{:}T|R \bullet P) \wedge (\exists x{:}T|R \bullet Q) \Rightarrow (\exists x{:}T|R \bullet P \wedge Q)$

≡"shunting"

$(\forall x{:}T|R \bullet P) \Rightarrow ((\exists x{:}T|R \bullet Q) \Rightarrow (\exists x{:}T|R \bullet P \wedge Q)$

⇐"∃/⇒, ⇒-right-mono"

$(\forall x{:}T|R \bullet P) \Rightarrow (\forall x{:}T|R \bullet Q \Rightarrow P \wedge Q)$

⇐"∀/⇒"

$(\forall x{:}T|R \bullet P \Rightarrow (Q \Rightarrow P \wedge Q))$

≡"shunting"

$(\forall x{:}T|R \bullet P \wedge Q \Rightarrow P \wedge Q)$❑

Theorem ∀-∃: $(\forall x{:}T|R \bullet P) \Rightarrow (\exists x{:}T|R \bullet P)$ if $(\exists x{:}T \bullet R \equiv True)$

Proof: In preceding theorem, instantiate Q with True. ❑

4.11 One-point and shifting

Theorem Δ-∃: $\Delta E \equiv (\exists x{:}T \bullet x \equiv E)$ where x fresh

Proof: Use bi-implication, as both sides proper.

(⇒) $\Delta E \Rightarrow (\exists x{:}T \bullet x \equiv E)$

≡"≡-refl, ∧-unit"

$(E \equiv E) \wedge \Delta E \Rightarrow (\exists x{:}T \bullet x \equiv E)$

≡"substitution"

$(x \equiv E)[x{:=}E] \wedge \Delta E \Rightarrow (\exists x{:}T \bullet x \equiv E)$

≡"instantiation"

True

(⇐) $(\exists x{:}T \bullet x \equiv E) \Rightarrow \Delta E$

≡"∃-lub"

$(\forall x{:}T \bullet (x \equiv E) \Rightarrow \Delta E)$

≡"⇒-subst"

$(\forall x{:}T \bullet (x \equiv E) \Rightarrow \Delta x)$

≡"constants proper"

True ❑

Theorem one-point: $(\forall x{:}T|x{\equiv}t \bullet P) \equiv \Delta t \Rightarrow P[x{:=}t]$ where x not free in t.

Proof:

$(\forall x{:}T|x{\equiv}t \bullet P)$

≡"trading"

$(\forall x{:}T \bullet x{\equiv}t \Rightarrow P)$

≡"⇒-subst"

$(\forall x{:}T \bullet x{\equiv}t \Rightarrow P[x{:=}t])$

≡"∃-lub, no x in P[x:=t]"

$(\exists x{:}T \bullet x{\equiv}t) \Rightarrow P[x{:=}t]$

≡"Δ-∃"

$\Delta t \Rightarrow P[x{:=}t]$ ❑

Theorem one-point: $(\exists x{:}T|x{\equiv}t \bullet P) \equiv P[x{:=}t] \wedge \Delta t$ where x not in t.

Theorem shifting: (i) $(\forall x{:}T|R \bullet P) \equiv (\forall x{:}T|R[x{:=}t] \bullet P[x{:=}t])$

(ii) $(\exists x{:}T|R \bullet P) \equiv (\exists x{:}T|R[x{:=}t] \bullet P[x{:=}t])$

provided that as a function of x, t is surjective and total, i.e.

(a) $(\forall y{:}T \bullet (\exists x{:}T \bullet y{\equiv}t))$, and

(b) $(\forall x{:}T \bullet (\exists y{:}T \bullet y{\equiv}t))$ (which is equivalent to $(\forall x{:}T \bullet \Delta t)$)

Proof of (i):

$(\forall x{:}T|R[x{:=}t] \bullet P[x{:=}t])$

≡"trading"

$(\forall x{:}T \bullet R[x{:=}t] \Rightarrow P[x{:=}t])$

≡"substitution property"

$(\forall x{:}T \bullet (R \Rightarrow P)[x{:=}t])$

It follows that we need only treat the case of no range.

Proof:

$(\forall x{:}T \bullet P[x{:=}t])$

≡"(b)"

$(\forall x{:}T \bullet P[x{:=}t]) \wedge (\forall x{:}T \bullet \Delta t)$

≡"∀/∧, absorption (appealing to ΔΔt), ∀/∧ again"

$(\forall x{:}T \bullet \neg\Delta t \vee P[x{:=}t])$

≡"renaming with y a fresh variable; let t' ≡ t[x:=y]"

$(\forall y{:}T \bullet \neg\Delta t' \vee P[x{:=}t'])$

≡"one point"

$(\forall y{:}T \bullet (\forall x{:}T|x{\equiv}t' \bullet P))$

≡"trading, interchange"

$(\forall x{:}T \bullet (\forall y{:}T \bullet x{\equiv}t' \Rightarrow P))$

≡"∃-lub , y not free in P"

$(\forall x{:}T \bullet (\exists y{:}T \bullet x \equiv t') \Rightarrow P)$

$\equiv$“(a), renaming”

$(\forall x{:}T \bullet P)$ ❑

Theorem instantiation: $(\forall x{:}T \bullet P) \wedge P[x{:=}t] \equiv (\forall x{:}T \bullet P)$ if Δt

Proof: One-point and range split. ❑

Theorem instantiation: $(\exists x{:}T \bullet P) \vee P[x{:=}t] \equiv (\exists x{:}T \bullet P)$ if Δt

Theorem $\forall$ strengthen: $(\forall x{:}T \bullet Q) \Rightarrow ((\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet Q \Rightarrow P))$

Proof:

$(\forall x{:}T \bullet Q) \Rightarrow ((\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet Q \Rightarrow P))$

$\equiv$“$\forall/\equiv$”

$(\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet P) \;\equiv\; (\forall x{:}T \bullet Q) \Rightarrow (\forall x{:}T \bullet Q \Rightarrow P)$

$\equiv$“$\Rightarrow/\forall$ twice”

$(\forall x{:}T \bullet (\forall x{:}T \bullet Q) \Rightarrow P) \;\equiv\; (\forall x{:}T \bullet (\forall x{:}T \bullet Q) \Rightarrow (Q \Rightarrow P))$

$\equiv$ “shunting”

$(\forall x{:}T \bullet (\forall x{:}T \bullet Q) \Rightarrow P) \;\equiv\; (\forall x{:}T \bullet (\forall x{:}T \bullet Q) \wedge Q \Rightarrow P)$

$\equiv$ “innstantiation using Δx”

$(\forall x{:}T \bullet (\forall x{:}T \bullet Q) \Rightarrow P) \;\equiv\; (\forall x{:}T \bullet (\forall x{:}T \bullet Q) \Rightarrow P)$ — $\equiv$-refl ❑

Theorem $\mathbb{B}$-instantiation: $(\forall x{:}\mathbb{B} \bullet P) \equiv P[x{:=}True] \wedge P[x{:=}False]$

Proof:

$(\forall x{:}\mathbb{B} \bullet P)$

$\equiv$“ constants proper, $\forall$ strengthen”

$(\forall x{:}\mathbb{B} \bullet \Delta x \Rightarrow P)$

$\equiv$“boolean properness”

$(\forall x{:}\mathbb{B} \bullet (x \equiv True) \vee (x \equiv False) \Rightarrow P)$

$\equiv$ “$\vee$-lub”

$(\forall x{:}\mathbb{B} \bullet (x \equiv True) \Rightarrow P) \wedge (\forall x{:}\mathbb{B} \bullet (x \equiv False) \Rightarrow P)$

$\equiv$ “trading and one-point twice using $\Delta True$ and $\Delta False$”

$P[x{:=}True] \wedge P[x{:=}False]$ ❑

Theorem $\mathbb{B}$-instantiation: $(\exists x{:}\mathbb{B} \bullet P) \equiv P[x{:=}True] \vee P[x{:=}False]$

4.12 Proper quantifications

Theorem $\Delta\forall$: $(\forall x{:}T \bullet \Delta P) \Rightarrow \Delta(\forall x{:}T \bullet P)$

Proof:

$(\forall x{:}T \bullet \Delta P)$

$\equiv$“$\Delta\mathbb{B}$”

$(\forall x{:}T \bullet (P \equiv True) \equiv P)$

$\Rightarrow$“$\forall/\equiv$”

$(\forall x{:}T \bullet P \equiv True) \equiv (\forall x{:}T \bullet P)$

$\equiv$ “$\forall$-truth”

$((\forall x{:}T \bullet P) \equiv True) \equiv (\forall x{:}T \bullet P)$

$\equiv$"$\Delta\mathbb{B}$"

$\Delta(\forall x{:}T \bullet P)$ ❑

Theorem $\Delta\exists$: $(\forall x{:}T \bullet \Delta P) \Rightarrow \Delta(\exists x{:}T \bullet P)$

5. Theorems of two- and three-valued logics

There are additional theorems common to three-valued logics; we present them here to avoid repetition To restrict the logic to at most three boolean values, we can postulate:

Postulate excluded fourth: $\Delta P \vee \Delta Q \vee (P \equiv Q)$

We will include the excluded fourth in E3, and it will turn out to be a theorem in **E**, **E3** and **EC**;. From it we derive the theorems that follow.

Theorem^{2,c,3} consistency: $(P \wedge Q \equiv P) \equiv (P \vee Q \equiv Q)$

Proof: Theorem already holds if ΔP or ΔQ. If $\neg(\Delta P \vee \Delta Q)$ then $P \equiv Q$ by excluded fourth, and the theorem follows easily. ❑

Theorem^{2,c,3} absorption: (i) $P \wedge (P \vee Q) \equiv P$

(ii) $P \vee (P \wedge Q) \equiv P$

Proof: If ΔP or ΔQ then proof is routine. Otherwise, conclude $P \equiv Q$ by excluded fourth. ❑

Theorem^{2,c,3}: $\Delta P \equiv (P \not\equiv \neg P)$

Proof: As both sides are proper, we can prove the equivalence by bi-implication. The left-to-right implication holds by an easy case analysis on P, and is done above. For the other direction:

$(P \not\equiv \neg P) \Rightarrow \Delta P$

$\equiv$"$\Rightarrow$-defn, $(P \equiv Q) \equiv ((P \not\equiv Q) \not\equiv \text{True})$"

$(P \equiv \neg P) \vee \Delta P$

$\equiv$"$\Delta P \equiv \Delta(\neg P)$"

$(P \equiv \neg P) \vee \Delta P \vee \Delta(\neg P)$

$\equiv$"excluded fourth"

True ❑

The following derived inference rule is useful in three-valued logics

truth cases $\dfrac{(P\equiv \text{True}) \equiv (Q\equiv \text{True}),\ (P\equiv \text{False}) \equiv (Q\equiv \text{False})}{P \equiv Q}$

It is justified by the following theorem, together with $\wedge$-intro and equanimity:

Theorem^{2,c,3} truth cases: $(P\equiv Q) \equiv ((P\equiv \text{True}) \equiv (Q\equiv \text{True})) \wedge ((P\equiv \text{False}) \equiv (Q\equiv \text{False}))$

Proof:

First we prove the lemma $((P\equiv \text{True}) \equiv (Q\equiv \text{True})) \equiv (P\equiv Q) \vee ((P\not\equiv \text{True}) \wedge (Q\not\equiv \text{True}))$

$(P \equiv Q) \vee ((P \not\equiv True) \wedge (Q \not\equiv True))$
≡"∨/∧, Δ(P≡Q)"
$((P \equiv Q) \vee (P \not\equiv True)) \wedge ((P \equiv Q) \vee (Q \not\equiv True))$
≡"∨-subst"
$((True \equiv Q) \vee (P \not\equiv True)) \wedge ((P \equiv True) \vee (Q \not\equiv True))$
≡"⇒-defn, all terms proper"
$((P \equiv True) \Rightarrow (True \equiv Q)) \wedge ((Q \equiv True) \Rightarrow (P \equiv True))$
≡"bi-implication, both sides proper; ≡-symm"
$(P \equiv True) \equiv (Q \equiv True)$

Similarly, $((P \equiv False) \equiv (Q \equiv False)) \equiv (P \equiv Q) \vee ((P \not\equiv False) \wedge (Q \not\equiv False))$

We prove the theorem itself using bi-implication as both sides are proper. The left-to-right implication is trivial. For the other direction:

$((P \equiv True) \equiv (Q \equiv True)) \wedge ((P \equiv False) \equiv (Q \equiv False)) \Rightarrow (P \equiv Q)$
≡"⇒-defn, antecedent proper, de Morgan"
$((P \equiv True) \not\equiv (Q \equiv True)) \vee ((P \equiv False) \not\equiv (Q \equiv False)) \vee (P \equiv Q)$
≡"above lemma, and de Morgan"
$((P \not\equiv Q) \wedge ((P \equiv True) \vee (Q \equiv True))) \vee ((P \not\equiv Q) \wedge ((P \equiv False) \vee (Q \equiv False))) \vee (P \equiv Q)$
≡"∧/∨, Δ(P≡Q)"
$((P \not\equiv Q) \wedge ((P \equiv True) \vee (Q \equiv True) \vee (P \equiv False) \vee (Q \equiv False)) \vee (P \equiv Q)$
≡"boolean properness"
$((P \not\equiv Q) \wedge (\Delta P \vee \Delta Q)) \vee (P \equiv Q)$
≡"∧/∨, Δ(P≡Q)"
$((P \not\equiv Q) \vee (P \equiv Q)) \wedge (\Delta P \vee \Delta Q \vee (P \equiv Q))$
≡"strong ≡"
$\Delta P \vee \Delta Q \vee (P \equiv Q)$ — excluded fourth ❑

Theorem^{2,c,3} ∧/∨: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
Proof: Use truth cases. For example,
$(P \wedge (Q \vee R) \equiv True) \equiv ((P \wedge Q) \vee (P \wedge R) \equiv True)$
reduces to
$P \equiv True \wedge (Q \equiv True \vee R \equiv True) \equiv ((P \equiv True \wedge Q \equiv True) \vee (P \equiv True \wedge R \equiv True))$
which holds from the previous version of ∧/∨ because Δ(P≡True). Similarly for the "≡ False" case. ❑

Theorem^{2,c,3} ∨/∧: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Theorem^{2,c,3} ∨/∀: $P \vee (\forall x{:}T|R \bullet Q) \equiv (\forall x{:}T|R \bullet P \vee Q)$ provided x does not occur free in P.
Proof : Use truth cases and the version of ∨/∀ that applies when the outside term is proper. ❑

Theorem^{2,c,3} ∧/∃: $P \wedge (\exists x{:}T|R \bullet Q) \equiv (\exists x{:}T|R \bullet P \wedge Q)$ provided x does not occur free in P.
Proof : Use truth cases and the version of ∧/∃ that applies when the outside term is proper. ❑

Theorem^{2,c,3} instantiation: $(\forall x{:}T \bullet P) \vee P[x{:=}t] \equiv P[x{:=}t]$ if Δt

Theorem^{2,c,3} instantiation: $(\exists x{:}T \bullet P) \wedge P[x{:=}t] \equiv P[x{:=}t]$ if Δt

6. E

We obtain the logic **E** by adding the postulate that all terms are proper

Axiom^{2} all proper: ΔE

The net effect is that in all the preceding theorems, all side-conditions requiring properness of subterms are vacuous. Note that the excluded fourth trivially holds, and so the theorems of the preceding section are part of **E**.

Theorem^{2} 2-valued: $(P\equiv True) \vee (P\equiv False)$

7. E3

The logic **E3** is $\mathbf{E}^{-}$ plus the axiom of the excluded fourth and an axiom defining $\bot_T$ for each type T:

Axiom^{3} $\neg\Delta\bot$: $\neg\Delta\bot_T$

Theorem^{3} $\bot$-improper: $(\forall x{:}T \bullet x\not\equiv\bot_T)$
Proof: Axiom variables defined. ❑

Theorem^{3} >2-valued: $\bot_{\mathbb{B}}\not\equiv True$ and $\bot_{\mathbb{B}}\not\equiv False$

Theorem^{3} 3-valued: $(P\equiv True) \vee (P\equiv False) \vee (P\equiv\bot_{\mathbb{B}})$
Proof:

$(P\equiv True) \vee (P\equiv False) \vee (P\equiv\bot_{\mathbb{B}})$
≡"boolean properness"
$\Delta P \vee (P\equiv\bot_{\mathbb{B}})$
≡"$\neg\Delta\bot$, $\vee$-unit"
$\Delta P \vee \Delta\bot_{\mathbb{B}} \vee (P\equiv\bot_{\mathbb{B}})$ — excluded fourth ❑

8. EC

For the logic **EC**, we introduce the binary infix operators $\sqcap$ and $\sqsubseteq$. If E and F are of type T, then so is $E\sqcap F$, while $E\sqsubseteq F$ is of type boolean. $E\sqcap F$ denotes the nondeterministic choice among E and F. $E\sqsubseteq F$ asserts that the possible outcomes of F are a subset of those of E. The axioms of **EC** are those of $\mathbf{E}^{-}$ plus the set of axioms that follow. We will deduce the excluded fourth; until we do, it is not used in proofs.

In the following axioms, E and F have type T:

Axiom^{c} $\sqsubseteq$-defn: $E\sqsubseteq F \equiv (\forall x{:}T \bullet F\sqsubseteq x \Rightarrow E\sqsubseteq x)$ x fresh
Axiom^{c} $\sqsubseteq$-antisymm: $E\sqsubseteq F \wedge F\sqsubseteq E \Rightarrow (E\equiv F)$

Axiom^c $\sqcap$-defn: $(\forall x{:}T \bullet E\sqcap F\sqsubseteq x \equiv E\sqsubseteq x \vee F\sqsubseteq x)$
Axiom^c $\mathbb{B}$-flat: $(\forall x,y{:}\mathbb{B} \bullet x\sqsubseteq y \equiv (x\equiv y))$

The theorems of **EC** follow.

Theorem^c $\Delta\sqsubseteq$: $\Delta(E\sqsubseteq F)$
Proof: Use $\sqsubseteq$-defn and $\Delta\forall$. ❑

Theorem^c $\sqsubseteq$-antisymm: $E\sqsubseteq F \wedge F\sqsubseteq E \equiv (E\equiv F)$
Proof: Use bi-implication as both sides proper. ❑

Theorem^c $\sqsubseteq$-refl: $E\sqsubseteq E$
Proof: Apply $\sqsubseteq$-defn. ❑

Theorem^c $\sqsubseteq$-trans: $E\sqsubseteq F \wedge F\sqsubseteq G \Rightarrow E\sqsubseteq G$
Proof: Apply $\sqsubseteq$-defn to the three subterms. ❑

Theorem^c $\equiv\sqsubseteq$: $(E\equiv F) \equiv (\forall x{:}T \bullet E\sqsubseteq x \equiv F\sqsubseteq x)$
Proof: Apply $\sqsubseteq$-antisymm and $\sqsubseteq$-defn to left-hand side. ❑

Theorem^c $\sqcap$-symm: $E\sqcap F \equiv F\sqcap E$
Proof: Apply $\equiv\sqsubseteq$ and use $\sqcap$-defn. ❑

Theorem^c $\sqcap$-idem: $E\sqcap E \equiv E$
Proof: Apply $\equiv\sqsubseteq$ and use $\sqcap$-defn. ❑

Theorem^c $\sqcap$-assoc: $E\sqcap(F\sqcap G) \equiv (E\sqcap F)\sqcap G$
Proof: Apply $\equiv\sqsubseteq$ and use $\sqcap$-defn. ❑

Theorem^c $\sqsubseteq\sqcap$: $E\sqsubseteq F \equiv (E\sqcap F\equiv E)$
Proof: Apply $\sqsubseteq$-defn to right-hand side and use $\sqcap$-defn . ❑

Theorem^c $\sqsubseteq$-lub: $E\sqsubseteq F\sqcap G \equiv E\sqsubseteq F \wedge E\sqsubseteq G$
Proof: Apply $\sqsubseteq$-defn to left-hand side and use $\sqcap$-defn . ❑

Theorem^c $\Delta/\sqcap$: $\Delta(E\sqcap F) \equiv (\Delta E \wedge E\sqsubseteq F) \vee (\Delta F \wedge F\sqsubseteq E)$
Proof:

$\Delta(E\sqcap F)$
≡"Δ-∃"
$(\exists x{:}T \bullet x \equiv E\sqcap F)$
≡"⊑-antisymm"
$(\exists x{:}T \bullet E\sqcap F\sqsubseteq x \wedge x\sqsubseteq E\sqcap F)$
≡"⊓-defn, ⊑-lub"
$(\exists x{:}T \bullet (E\sqsubseteq x \vee F\sqsubseteq x) \wedge x\sqsubseteq E \wedge x\sqsubseteq F)$
≡"∧/∨ using Δ⊑"

$(\exists x{:}T \bullet (E \sqsubseteq x \wedge x \sqsubseteq E \wedge x \sqsubseteq F) \vee (F \sqsubseteq x \wedge x \sqsubseteq E \wedge x \sqsubseteq F))$

$\equiv$“$\sqsubseteq$-antisymm twice”

$(\exists x{:}T \bullet ((x \equiv E) \wedge x \sqsubseteq F) \vee ((x \equiv F) \wedge x \sqsubseteq E))$

$\equiv$“$\wedge$-subst twice”

$(\exists x{:}T \bullet ((x \equiv E) \wedge E \sqsubseteq F) \vee ((x \equiv F) \wedge F \sqsubseteq E))$

$\equiv$“$\exists/\vee$”

$(\exists x{:}T \bullet (x \equiv E) \wedge E \sqsubseteq F) \vee (\exists x{:}T \bullet (x \equiv F) \wedge F \sqsubseteq E)$

$\equiv$“$\wedge/\exists$ (twice) using x not in E or F and $\Delta\sqsubseteq$”

$((\exists x{:}T \bullet x \equiv E) \wedge E \sqsubseteq F) \vee ((\exists x{:}T \bullet x \equiv F) \wedge F \sqsubseteq E)$

$\equiv$“Δ-$\exists$ twice”

$(\Delta E \wedge E \sqsubseteq F) \vee (\Delta F \wedge F \sqsubseteq E)$ ❑

Theorem^c >2-valued: (i) $(True \sqcap False \not\equiv True)$
(ii) $(True \sqcap False \not\equiv False)$

Proof (i):

$True \sqcap False \not\equiv True$

$\equiv$“$\sqsubseteq\sqcap$”

$\neg(True \sqsubseteq False)$

$\equiv$“$\mathbb{B}$-flat instantiated”

$\neg(True \equiv False)$ — two values ❑

Theorem^c $\equiv True \sqcap False$: $(P \equiv True \sqcap False) \equiv P \sqsubseteq True \wedge P \sqsubseteq False$

Proof:

$P \equiv True \sqcap False$

$\equiv$“$\equiv\sqsubseteq$”

$(\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \equiv True \sqcap False \sqsubseteq x)$

$\equiv$“$\mathbb{B}$-instantiation”

$(P \sqsubseteq True \equiv True \sqcap False \sqsubseteq True) \wedge (P \sqsubseteq False \equiv True \sqcap False \sqsubseteq False)$

$\equiv$“$\sqcap$-defn, $\equiv$-truth, $\not\equiv$-truth”

$(P \sqsubseteq True) \wedge (P \sqsubseteq False)$ ❑

We now introduce the axiom of the excluded miracle:

Axiom^c no-miracles: $(\exists x{:}T \bullet E \sqsubseteq x)$ where E has type T and x fresh

Theorem^c $\sqsubseteq True$: $P \sqsubseteq True \equiv (P \not\equiv False)$

Proof:

$P \sqsubseteq True$

$\equiv$“no-miracles”

$(\exists x{:}\mathbb{B} \bullet P \sqsubseteq x) \Rightarrow P \sqsubseteq True$

$\equiv$“$\exists$-lub”

$(\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \Rightarrow P \sqsubseteq True))$

$\equiv$“$\mathbb{B}$-instantiation”

$(P \sqsubseteq True \Rightarrow P \sqsubseteq True) \wedge (P \sqsubseteq False \Rightarrow P \sqsubseteq True)$

$\equiv$“$\Rightarrow$-refl, $\wedge$-unit”

$P \sqsubseteq False \Rightarrow P \sqsubseteq True$

$\equiv$"$\Rightarrow$-defn using $\Delta\sqsubseteq$, de Morgan"

$\neg(P\sqsubseteq \text{False} \wedge \neg(P\sqsubseteq \text{True}))$

$\equiv$" $\Delta\sqsubseteq$ and elementary properties of $\equiv$ and $\neg$"

$\neg((P\sqsubseteq \text{False} \equiv \text{True}) \wedge (P\sqsubseteq \text{True} \equiv \text{False}))$

$\equiv$"$\sqsubseteq$-refl, $\mathbb{B}$-flat"

$\neg((P\sqsubseteq \text{False} \equiv \text{False}\sqsubseteq \text{False}) \wedge (P\sqsubseteq \text{True} \equiv \text{False}\sqsubseteq \text{True}))$

$\equiv$"$\mathbb{B}$-instantiation"

$\neg(\forall x{:}\mathbb{B} \bullet P\sqsubseteq x \equiv \text{False}\sqsubseteq x)$

$\equiv$"$\equiv\sqsubseteq$"

$\neg(P\equiv \text{False})$ ❑

Theorem^c $\sqsubseteq$False: $P\sqsubseteq \text{False} \equiv (P\not\equiv \text{True})$

Theorem^c $\sqcap$True: $(P\sqcap Q\equiv \text{True}) \equiv (P\equiv \text{True}) \wedge (Q\equiv \text{True})$

Proof:

$P\sqcap Q\equiv \text{True}$

$\equiv$"$\equiv\sqsubseteq$"

$(\forall x{:}\mathbb{B} \bullet P\sqcap Q\sqsubseteq x \equiv \text{True}\sqsubseteq x)$

$\equiv$"$\mathbb{B}$-instantiation"

$(P\sqcap Q\sqsubseteq \text{True} \equiv \text{True}\sqsubseteq \text{True}) \wedge (P\sqcap Q\sqsubseteq \text{False} \equiv \text{True}\sqsubseteq \text{False})$

$\equiv$"$\sqcap$-symm, $\mathbb{B}$-flat, $\Delta\sqsubseteq$ and elementary properties of $\equiv$ and $\neg$"

$P\sqcap Q\sqsubseteq \text{True} \wedge \neg(P\sqcap Q\sqsubseteq \text{False})$

$\equiv$"$\sqcap$-defn, de Morgan"

$P\sqcap Q\sqsubseteq \text{True} \wedge \neg(P\sqsubseteq \text{False}) \wedge \neg(Q\sqsubseteq \text{False})$

$\equiv$"$\sqsubseteq$False twice"

$P\sqcap Q\sqsubseteq \text{True} \wedge (P\equiv \text{True}) \wedge (Q\equiv \text{True})$

$\equiv$"$\wedge$-subst twice, $\sqcap$-idem , and $\sqsubseteq$-refl"

$(P\equiv \text{True}) \wedge (Q\equiv \text{True})$ ❑

Theorem^c $\sqcap$False: $(P\sqcap Q\equiv \text{False}) \equiv (P\equiv \text{False}) \wedge (Q\equiv \text{False})$

Theorem^c $\sqsubseteq\mathbb{B}$: $P\sqsubseteq Q \equiv \neg\Delta P \vee (P\equiv Q)$

Proof:

$P\sqsubseteq Q$

$\equiv$"$\sqsubseteq$-defn"

$(\forall x{:}\mathbb{B} \bullet Q\sqsubseteq x \Rightarrow P\sqsubseteq x)$

$\equiv$"$\mathbb{B}$-instantiation"

$(Q\sqsubseteq \text{True} \Rightarrow P\sqsubseteq \text{True}) \wedge (Q\sqsubseteq \text{False} \Rightarrow P\sqsubseteq \text{False})$

$\equiv$"$\sqsubseteq$True, $\sqsubseteq$False"

$((Q\not\equiv \text{False}) \Rightarrow (P\not\equiv \text{False})) \wedge ((Q\not\equiv \text{True}) \Rightarrow (P\not\equiv \text{True}))$

$\equiv$"$\Rightarrow$-defn twice using $\not\equiv$-truth"

$((Q\equiv \text{False}) \vee (P\not\equiv \text{False})) \wedge ((Q\equiv \text{True}) \vee (P\not\equiv \text{True}))$

$\equiv$"$\vee$-subst twice"

$((Q\equiv P) \vee (P\not\equiv \text{False})) \wedge ((Q\equiv P) \vee (P\not\equiv \text{True}))$

$\equiv$"$\vee/\wedge$ using $\equiv$-truth, $\equiv$-symm twice"

$(P\equiv Q) \vee ((P\not\equiv \text{False})) \wedge (P\not\equiv \text{True}))$

≡"de Morgan and boolean properness"
$(P\equiv Q) \vee \neg\Delta P$ ❑

Theorem^c ⊑-exchange: $\neg P\sqsubseteq Q \equiv P\sqsubseteq\neg Q$
Proof: Apply ⊑𝔹 to one side. ❑

Theorem^c Δ/⊓𝔹: $\Delta(P\sqcap Q) \equiv \Delta P \wedge (P\equiv Q)$
Proof:

$\Delta(P\sqcap Q)$
≡"Δ/⊓"
$(\Delta P \wedge P\sqsubseteq Q) \vee (\Delta Q \wedge Q\sqsubseteq P)$
≡"⊑𝔹, and elementary properties of ∧ and ∨"
$(\Delta P \wedge (P\equiv Q)) \vee (\Delta Q \wedge (Q\equiv P))$
≡"≡-symm and ∧/∨ using ≡-truth"
$(\Delta P \vee \Delta Q) \wedge (P\equiv Q)$
≡"∧-subst, ∨-idem"
$\Delta P \wedge (P\equiv Q)$ ❑

Theorem^c excluded fourth: $\Delta P \vee \Delta Q \vee (P\equiv Q)$
Proof:

$\Delta P \vee \Delta Q \vee (P\equiv Q)$
≡"⊑-antisymm"
$\Delta P \vee \Delta Q \vee (P\sqsubseteq Q \wedge Q\sqsubseteq P)$
≡ "∨/∧ using Δ⊑"
$(\Delta P \vee \Delta Q \vee P\sqsubseteq Q) \wedge (\Delta P \vee \Delta Q \vee Q\sqsubseteq P)$
≡ "⊑𝔹"
$(\Delta P \vee \Delta Q \vee \neg\Delta P \vee (P\equiv Q)) \wedge (\Delta P \vee \Delta Q \vee \neg\Delta Q \vee (Q\equiv P))$
≡ "excluded middle twice using ΔΔ, and elementary properties of ∨ and ∧"
True ❑

Theorem^c ¬/⊓: $\neg(P\sqcap Q) \equiv \neg P\sqcap\neg Q$
Proof: Prove that each side refines the other using, in each case, theorems ⊑-exchange and ⊑-lub. ❑

Theorem^c ∨/⊓: $P\vee(Q\sqcap R) \equiv (P\vee Q) \sqcap (P\vee R)$ and $(P\sqcap Q)\vee R \equiv (P\vee R) \sqcap (Q\vee R)$
Proof: Truth cases. ❑

Theorem^c ∧/⊓: $P\wedge(Q\sqcap R) \equiv (P\wedge Q) \sqcap (P\wedge R)$ and $(P\sqcap Q)\wedge R \equiv (P\wedge R) \sqcap (Q\wedge R)$

Theorem^c ⇒/⊓: $P\Rightarrow(Q\sqcap R) \equiv (P\Rightarrow Q) \sqcap (P\Rightarrow R)$

Theorem^c ⊓/∧: $P\sqcap(Q\wedge R) \equiv (P\sqcap Q) \wedge (P\sqcap R)$ and $(P\wedge Q) \sqcap R \equiv (P\sqcap R) \wedge (Q\sqcap R)$
Proof: Truth cases. ❑

Theorem^c ¬⊑-mono: $P\sqsubseteq Q \Rightarrow \neg P\sqsubseteq\neg Q$

Theorem^c ∨⊑-mono: $P\sqsubseteq Q \Rightarrow (P\vee R)\sqsubseteq(Q\vee R)$ and $Q\sqsubseteq R \Rightarrow (P\vee Q)\sqsubseteq(P\vee R)$

Theorem^c ∧⊑-mono: P⊑Q ⇒ (P∧R)⊑(Q∧R) and Q⊑R ⇒ (P∧Q)⊑(P∧R)

Theorem^c ⊓/∨: P⊓(Q∨R) ≡ (P⊓Q) ∨ (P⊓R) and (P∨Q)⊓R ≡ (P⊓R) ∨ (Q⊓R)

Theorem^c ⊓/∀: (∀x:T • P⊓Q) ≡ P⊓(∀x:T • Q) if x does not occur free in P.
Proof: Truth cases. ❑

Theorem^c ⊓/∃: (∃x:T • P⊓Q) ≡ P⊓(∃x:T • Q) if x does not occur free in P.

Theorem^c ∀⊑True: (∀x:T • P) ⊑ True ≡ (∀x:T • P ⊑ True)

Theorem^c ∀⊑False: (∀x:T • P) ⊑ False ≡ (∃x:T • P ⊑ False)

Theorem^c ∀/⊑: (∀x:T • P⊑Q) ⇒ ((∀x:T • P) ⊑ (∀x:T • Q))
Proof: Working on the consequent, apply ⊑-defn and use preceding two theorems. ❑

Theorem^c 3-valued: (P≡True) ∨ (P≡False) ∨ (P≡True⊓False)
Proof:

(P≡True) ∨ (P≡False) ∨ (P≡True⊓False)
≡"≡True⊓False"
(P≡True) ∨ (P≡False) ∨ (P⊑True ∧ P⊑False)
≡"⊑True, ⊑False"
(P≡True) ∨ (P≡False) ∨ ((P≢False) ∧ (P≢True))
≡"de Morgan, excluded middle using ≡-truth"
True ❑

9. EB

The logic **EB** uses the operators ⊓ and ⊑ governed by the same axioms as in **EC**:

Axiom^b ⊑-defn: E⊑F ≡ (∀x:T • F⊑x ⇒ E⊑x) x fresh
Axiom^b ⊑-antisymm: E⊑F ∧ F⊑E ⇒ (E≡F)
Axiom^b ⊓-defn: (∀x:T • E⊓F ⊑ x ≡ E⊑x ∨ F⊑x)
Axiom^b 𝔹-flat: (∀x,y:𝔹 • x⊑y ≡ (x≡y))

(The choice symbol ⊓ is written as a comma in [7] to conform with standard practice in bunch theory.) The axioms of EC are the above and those of $\mathbf{E}^{-}$, together with four more we shall introduce later. We immediately inherit the following theorems from **EC**, and their proofs:

Theorem^b Δ⊑: Δ(E⊑F)
Theorem^b ⊑-antisymm: E⊑F ∧ F⊑E ≡ (E≡F)
Theorem^b ⊑-refl: E⊑E
Theorem^b ⊑-trans: E⊑F ∧ F⊑G ⇒ E⊑G

Theorem^b ≡⊑: (E≡F) ≡ (∀x:T • E⊑x ≡ F⊑x)
Theorem^b ⊓-symm: E⊓F ≡ F⊓E
Theorem^b ⊓-idem: E⊓E ≡ E
Theorem^b ⊓-assoc: E⊓(F⊓G) ≡ (E⊓F)⊓G
Theorem^b ⊑⊓: E⊑F ≡ (E⊓F≡E)
Theorem^b ⊑-lub: E⊑F⊓G ≡ E⊑F ∧ E⊑G
Theorem^b Δ/⊓: Δ(E⊓F) ≡ (ΔE ∧ E⊑F) ∨ (ΔF ∧ F⊑E)
Theorem^b >2-valued: (True⊓False≢True) and (True⊓False≢False)
Theorem^b ≡True⊓False: (P≡True⊓False) ≡ P⊑True ∧ P⊑False

EB differs from **EC** in including a miracle value called $null_T$ for each type T:

Axiom^b null: (∀x:T • ¬($null_T$ ⊑ x))

We may omit the subscript in $null_T$ when it is evident from context or when its value is not significant.

Theorem^b null-max: E ⊑ null
Proof: Apply ⊑-defn. ❑

Theorem^b ⊓-unit: null⊓E ≡ E
Proof: Apply ⊑⊓. ❑

Theorem^b null-max: null ⊑ E ≡ (E≡null)

Theorem^b is-null: (E≡null) ≡ ¬(∃x:T • E⊑x) x fresh
Proof: Apply ≡⊑ to left-hand side. ❑

Theorem^b ¬Δnull: ¬Δnull
Proof:

¬Δnull
≡“Δ-∃”
¬(∃x:T • null≡x)
≡“ null-max”
¬(∃x:T • null⊑x)
≡“de Morgan”
(∀x:T • ¬(null⊑x)) — null ❑

Theorem^b ⊓null: (E⊓F ≡ null) ≡ (E ≡ null) ∧ (F ≡ null)
Proof: Apply ≡⊑ to left-hand side. ❑

Theorem^b null-new: (i) True≢$null_{\mathbb{B}}$ (ii) False≢$null_{\mathbb{B}}$
Proof (i): True⊑True but ¬ ($null_{\mathbb{B}}$⊑True) ❑

Theorem^b ⊑True: P⊑True ≡ (P≢False) ∧ (P≢$null_{\mathbb{B}}$)

Proof:

$P \sqsubseteq True$

$\equiv$"$\equiv$-truth, $\Rightarrow$-left-unit"

$((P \equiv null_{\mathbb{B}}) \vee (P \not\equiv null_{\mathbb{B}})) \Rightarrow P \sqsubseteq True$

$\equiv$"$\vee$-lub"

$((P \equiv null_{\mathbb{B}}) \Rightarrow P \sqsubseteq True) \wedge ((P \not\equiv null_{\mathbb{B}}) \Rightarrow P \sqsubseteq True)$

$\equiv$"$\Rightarrow$-defn and elementary properties of $\equiv$"

$((P \not\equiv null_{\mathbb{B}}) \vee P \sqsubseteq True) \wedge ((P \not\equiv null_{\mathbb{B}}) \Rightarrow P \sqsubseteq True)$

$\equiv$"$\vee$-subst"

$((P \not\equiv null_{\mathbb{B}}) \vee null_{\mathbb{B}} \sqsubseteq True) \wedge ((P \not\equiv null_{\mathbb{B}}) \Rightarrow P \sqsubseteq True)$

$\equiv$"null-max, $\vee$–unit"

$(P \not\equiv null_{\mathbb{B}}) \wedge ((P \not\equiv null_{\mathbb{B}}) \Rightarrow P \sqsubseteq True)$

$\equiv$"is-null"

$(P \not\equiv null_{\mathbb{B}}) \wedge ((\exists x{:}\mathbb{B} \bullet P \sqsubseteq x) \Rightarrow P \sqsubseteq True)$

$\equiv$"see proof of for $\sqsubseteq$True in EC"

$(P \not\equiv null_{\mathbb{B}}) \wedge (P \not\equiv False)$ ❑

Theorem^b $\sqsubseteq$False: $P \sqsubseteq False \equiv (P \not\equiv True) \wedge (P \not\equiv null_{\mathbb{B}})$

Theorem^b $\equiv null_{\mathbb{B}}$: $P \not\equiv null_{\mathbb{B}} \equiv P \sqsubseteq True \vee P \sqsubseteq False$

Proof: Apply $\sqsubseteq$True and $\sqsubseteq$False to right-hand side. ❑

Theorem^b 4-valued: $(P \equiv True) \vee (P \equiv False) \vee (P \equiv null_{\mathbb{B}}) \vee (P \equiv True \sqcap False)$

Proof:

$(P \equiv True) \vee (P \equiv False) \vee (P \equiv null_{\mathbb{B}}) \vee (P \equiv True \sqcap False)$

$\equiv$"$\vee$-idem"

$((P \equiv True) \vee (P \equiv null_{\mathbb{B}})) \vee ((P \equiv False) \vee (P \equiv null_{\mathbb{B}})) \vee (P \equiv True \sqcap False)$

$\equiv$"$\sqsubseteq$True, $\sqsubseteq$False, de Morgan"

$\neg (P \sqsubseteq True) \vee \neg(P \sqsubseteq False) \vee (P \equiv True \sqcap False)$

$\equiv$"$\equiv True \sqcap False$"

$\neg (P \sqsubseteq True) \vee \neg(P \sqsubseteq False) \vee (P \sqsubseteq True \wedge P \sqsubseteq False)$

$\equiv$"de Morgan, excluded middle using $\Delta\sqsubseteq$"

True ❑

The behaviour of the boolean operators with respect to $null_{\mathbb{B}}$ is fixed by the following two axioms:

Axiom^b $\neg$null: $\neg null_{\mathbb{B}} \equiv null_{\mathbb{B}}$

Axiom^b $\vee$null: $True \sqcap False \vee null_{\mathbb{B}} \equiv null_{\mathbb{B}}$

Theorem^b $\sqsubseteq$-exchange: $\neg P \sqsubseteq Q \equiv P \sqsubseteq \neg Q$

Proof: First show $\neg P \sqsubseteq True \equiv P \sqsubseteq False$ by applying $\sqsubseteq$True to the left-hand side and appealing to axiom exchange. Similarly show $\neg P \sqsubseteq False \equiv P \sqsubseteq True$.

$\neg P \sqsubseteq Q$

$\equiv$"$\sqsubseteq$-defn"

$(\forall x,y{:}\mathbb{B} \bullet Q \sqsubseteq x \Rightarrow \neg P \sqsubseteq x)$

$\equiv$"$\mathbb{B}$-instantiation"
$(Q \sqsubseteq True \Rightarrow \neg P \sqsubseteq True) \wedge (Q \sqsubseteq False \Rightarrow \neg P \sqsubseteq False)$
$\equiv$"lemmas above"
$(\neg Q \sqsubseteq False \Rightarrow P \sqsubseteq False) \wedge (\neg Q \sqsubseteq True \Rightarrow P \sqsubseteq True)$
$\equiv$"$\mathbb{B}$-instantiation and $\sqsubseteq$-defn"
$P \sqsubseteq \neg Q$ ❑

Theorem^b $\neg/\sqcap$: $\neg(P \sqcap Q) \equiv \neg P \sqcap \neg Q$
Proof: Prove that each side refines the other using, in each case, theorems $\sqsubseteq$-exchange and $\sqsubseteq$-lub. ❑

Theorem^b $\neg/\sqcap$: $\neg(True \sqcap False) \equiv True \sqcap False$

Theorem^b $\wedge$null: $True \sqcap False \wedge null_{\mathbb{B}} \equiv null_{\mathbb{B}}$
Proof: Use $\wedge$-defn and axiom $\vee$null. ❑

Note that $\vee$ does not distribute over $\sqcap$. In particular, $True \sqcap False \vee null_{\mathbb{B}}$ is axiomatically equivalent to $null_{\mathbb{B}}$ whereas $(True \vee null_{\mathbb{B}}) \sqcap (False \vee null_{\mathbb{B}})$ reduces to $True \sqcap null_{\mathbb{B}}$ which reduces to True. Apart from this case (and its trivial variations), $\vee$ distributes over $\sqcap$. Similar remarks hold for $\wedge$. Concomitantly, conjunction and disjunction are not monotonic with respect to $\sqsubseteq$ (the one case where disjunction fails to be montonic is $T \sqcap F \sqsubseteq T$ but not $(T \sqcap F \vee null_{\mathbb{B}}) \sqsubseteq (T \vee null_{\mathbb{B}})$).

The behaviour of the quantifiers with respect to $null_{\mathbb{B}}$ is governed by the following axiom:

Axiom^b $\exists$null: $((\exists x{:}T \bullet P) \equiv null_{\mathbb{B}}) \equiv (\exists x{:}T \bullet P \equiv null_{\mathbb{B}}) \wedge (\forall x{:}T \bullet P \not\equiv True)$

Theorem^b $\exists \sqsubseteq$False: $(\exists x{:}T \bullet P) \sqsubseteq False \equiv (\forall x{:}T \bullet P \sqsubseteq False)$
Proof:
$(\exists x{:}T \bullet P) \sqsubseteq False$
$\equiv$"$\sqsubseteq$False"
$((\exists x{:}T \bullet P) \not\equiv True) \wedge ((\exists x{:}T \bullet P) \not\equiv null_{\mathbb{B}})$
$\equiv$"$\exists$-non-truth, $\exists$null, de Morgan"
$(\forall x{:}T \bullet P \not\equiv True) \wedge ((\forall x{:}T \bullet P \not\equiv null_{\mathbb{B}}) \vee \neg(\forall x{:}T \bullet P \not\equiv True))$
$\equiv$"absorption using $\Delta\forall$ and $\not\equiv$-truth"
$(\forall x{:}T \bullet P \not\equiv True) \wedge (\forall x{:}T \bullet P \not\equiv null_{\mathbb{B}})$
$\equiv$"$\forall/\wedge$"
$(\forall x{:}T \bullet (P \not\equiv True) \wedge (P \not\equiv null_{\mathbb{B}}))$
$\equiv$"$\sqsubseteq$False"
$(\forall x{:}T \bullet P \sqsubseteq False)$ ❑

Theorem^b $\exists \sqsubseteq$True: $(\exists x{:}T \bullet P) \sqsubseteq True \equiv (\exists x{:}T \bullet P \equiv True) \vee ((\exists x{:}T \bullet P \sqsubseteq True) \wedge (\forall x{:}T \bullet P \not\equiv null_{\mathbb{B}}))$

Theorem^b $\forall$null: $((\forall x{:}T \bullet P) \equiv null_{\mathbb{B}}) \equiv (\exists x{:}T \bullet P \equiv null_{\mathbb{B}}) \wedge (\forall x{:}T \bullet P \not\equiv False)$

Theorem^b $\forall \sqsubseteq$True: $(\forall x{:}T \bullet P) \sqsubseteq True \equiv (\forall x{:}T \bullet P \sqsubseteq True)$

Theorem[b] ∀⊑False: $(\forall x{:}T \bullet P) \sqsubseteq \text{False} \equiv (\exists x{:}T \bullet P\equiv\text{False}) \vee ((\exists x{:}T \bullet P\sqsubseteq\text{False}) \wedge (\forall x{:}T \bullet P\not\equiv\text{null}_{\mathbb{B}}))$

10. E4

The logic **E4** includes the binary infix operators ⊓ and ⊑ of **EC**, as well as $\perp_T$ of **E3**. We may omit the subscript in $\perp_T$ when it is not significant or can be inferred from context. As a convenience, we introduce the prefix operator τ, where τE by definition abbreviates $E\not\equiv\perp$. The axioms of **E4** are those of $\mathbf{E}^{-}$ plus the various axioms given below. In the following axioms, E and F have type T:

Axiom[4] ⊑-defn: $E\sqsubseteq F \equiv (\tau E\Rightarrow\tau F) \wedge (\forall x{:}T \bullet F\sqsubseteq x \Rightarrow E\sqsubseteq x)$ x fresh
Axiom[4] ⊑-antisymm: $E\sqsubseteq F \wedge F\sqsubseteq E \Rightarrow (E\equiv F)$
Axiom[4] ⊓-defn: $(\forall x{:}T \bullet E\sqcap F \sqsubseteq x \equiv E\sqsubseteq x \vee F\sqsubseteq x)$
Axiom[4] 𝔹-flat: $(\forall x,y{:}\mathbb{B} \bullet x\sqsubseteq y \equiv (x\equiv y))$
Axiom[4] τ-defn: $\tau E \equiv (E\not\equiv\perp_T)$
Axiom[4] τ/⊓: $\tau(E\sqcap F) \equiv \tau E \wedge\tau F$

Observe that apart from a simple extra conjunct in axiom ⊑-defn, these are also axioms of **EC**, and hence we inherit the associated theorems of **EC** with just minor amendments.

Theorem[4] Δ⊑: $\Delta(E\sqsubseteq F)$
Proof: Use ⊑-defn, τ-defn, and Δ∀. ❑

Theorem[4] ⊑-antisymm: $E\sqsubseteq F \wedge F\sqsubseteq E \equiv (E\equiv F)$
Proof: Use bi-implication as both sides proper. ❑

Theorem[4] ⊑-refl: $E\sqsubseteq E$
Proof: Apply ⊑-defn. ❑

Theorem[4] ⊑-trans: $E\sqsubseteq F \wedge F\sqsubseteq G \Rightarrow E\sqsubseteq G$
Proof: Apply ⊑-defn to the three subterms. ❑

Theorem[4] ≡⊑: $(E\equiv F) \equiv (\tau E\equiv\tau F) \wedge (\forall x{:}T \bullet E\sqsubseteq x \equiv F\sqsubseteq x)$
Proof: Apply ⊑-antisymm and ⊑-defn to left-hand side. ❑

Theorem[4] ⊓-symm: $E\sqcap F \equiv F\sqcap E$
Proof: Apply ≡⊑, τ/⊓, and ⊓-defn. ❑

Theorem[4] ⊓-idem: $E\sqcap E \equiv E$
Proof: Apply ≡⊑ and ⊓-defn. ❑

Theorem[4] ⊓-assoc: $E\sqcap(F\sqcap G) \equiv (E\sqcap F)\sqcap G$
Proof: Apply ≡⊑ and ⊓-defn. ❑

Theorem[4] ⊑⊓: E⊑F ≡ (E⊓F≡E)
Proof: Apply ⊑-defn and ⊓-defn to the right-hand side. ❑

Theorem [4] ⊓-zero: ⊥⊓E ≡ ⊥

Theorem[4] ⊑-lub: E⊑F⊓G ≡ E⊑F ∧ E⊑G
Proof: Apply ⊑-defn and ⊓-defn to the left-hand side. ❑

Theorem[4] Δ/⊓: Δ(E⊓F) ≡ (ΔE ∧ E⊑F) ∨ (ΔF ∧ F⊑E)
Proof: Same as in **EC**. ❑

Theorem[4] >2-valued: (i) (True⊓False≢True)
(ii) (True⊓False≢False)
Proof: Same as in **EC**. ❑

Theorem[4] Δτ: ΔτE

Theorem[4] ⊥⊑: ⊥⊑E
Proof:

⊥⊑E
≡"⊑⊓"
⊥⊓E≡⊥
≡"τ-defn"
¬τ(⊥⊓E)
≡"τ/⊓"
¬(τ⊥ ∧ τE)
≡"¬τ⊥"
True ❑

Theorem[4] ⊑⊥: E⊑⊥ ≡ (E≡⊥)
Proof: Apply ⊑-antisymm and ⊥⊑ to right-hand side. ❑

We now introduce axioms that ensure that ⊥ is distinct from constants:

Axiom[4] Δ⇒τ: ΔE⇒τE

Theorem[4] ¬Δ⊥: $\neg\Delta\bot_T$

Theorem[4] τx: (∀x:T • τx)
Proof: (∀x:T • Δx) and Δ⇒τ: ❑

Theorem[4] ⊥-unique (i) ⊥≢True (ii) ⊥≢False (iii) ⊥≢True⊓False
Proof: Apply Δ⇒τ to (i) and (ii), and to τ/⊓ (iii). ❑

Theorem[4] ≡True⊓False: (P≡True⊓False) ≡ τP ∧ (P⊑True) ∧ (P⊑False)
Proof:

$P \equiv True \sqcap False$

$\equiv$ "$\equiv\sqsubseteq$"

$(\tau P \equiv \tau(True \sqcap False)) \wedge (\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \equiv True \sqcap False \sqsubseteq x)$

$\equiv$ "$\tau/\sqcap$, τx, $\Delta\tau$"

$\tau P \wedge (\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \equiv True \sqcap False \sqsubseteq x)$

$\equiv$ "$\mathbb{B}$-instantiation"

$\tau P \wedge (P \sqsubseteq True \equiv True \sqcap False \sqsubseteq True) \wedge (P \sqsubseteq False \equiv True \sqcap False \sqsubseteq False)$

$\equiv$ "elementary properties of $\sqsubseteq$ and $\equiv$"

$\tau P \wedge P \sqsubseteq True \wedge P \sqsubseteq False$ ❑

E4 includes the axiom of the excluded miracle:

Axiom^4 no-miracles: $(\exists x{:}T \bullet E \sqsubseteq x)$ where E has type T and x fresh

Theorem^4 $\sqsubseteq$True: $P \sqsubseteq True \equiv (P \not\equiv False)$

Proof: If $\neg\tau P$ then the result is trivial. Assume τP.

$P \sqsubseteq True$

$\equiv$ "exactly as in the proof in **EC**"

$\neg(\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \equiv False \sqsubseteq x)$

$\equiv$ "assuming τP, τFalse, $\wedge$-unit"

$\neg((\tau P \equiv \tau False) \wedge (\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \equiv False \sqsubseteq x)$

$\equiv$ "$\equiv\sqsubseteq$"

$\neg (P \equiv False)$ ❑

Theorem^4 $\sqsubseteq$False: $P \sqsubseteq False \equiv (P \not\equiv True)$

Theorem^4 $\sqcap$True: $(P \sqcap Q \equiv True) \equiv (P \equiv True) \wedge (Q \equiv True)$

Proof:

$P \sqcap Q \equiv True$

$\equiv$ "$\equiv\sqsubseteq$"

$(\tau(P \sqcap Q) \equiv \tau True) \wedge (\forall x{:}\mathbb{B} \bullet P \sqcap Q \sqsubseteq x \equiv True \sqsubseteq x)$

$\equiv$ "$\tau/\sqcap$, τx, $\Delta\tau$"

$\tau P \wedge \tau Q \wedge (\forall x{:}\mathbb{B} \bullet P \sqcap Q \sqsubseteq x \equiv True \sqsubseteq x)$

$\equiv$ "as in proof in **EC**"

$\tau P \wedge \tau Q \wedge (P \equiv True) \wedge (Q \equiv True)$

$\equiv$ "$\wedge$-subst twice and elementary properties of $\wedge$"

$(P \equiv True) \wedge (Q \equiv True)$ ❑

Theorem^4 $\sqcap$False: $(P \sqcap Q \equiv False) \equiv (P \equiv False) \wedge (Q \equiv False)$

Theorem^4 $\mathbb{B}$-flat: $(\forall x{:}\mathbb{B} \bullet x \sqsubseteq P \equiv (x \equiv P))$

Proof: Appealing to bi-implication and $\sqsubseteq$-antisymm we need only prove $x \sqsubseteq P \Rightarrow (P \sqsubseteq x)$.

$x \sqsubseteq P$

$\equiv$ "$\sqsubseteq$-defn"

$(\tau x \Rightarrow \tau P) \wedge (\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow x \sqsubseteq y)$

$\Rightarrow$ "weakening"

$(\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow x \sqsubseteq y)$
$\equiv$"$\Rightarrow/\wedge$, $\Rightarrow$-refl, $\wedge$-unit"
$(\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow P \sqsubseteq y \wedge x \sqsubseteq y)$
$\equiv$"$\mathbb{B}$-flat"
$(\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow P \sqsubseteq y \wedge (x \equiv y))$
$\equiv$"$\wedge$-subst"
$(\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow P \sqsubseteq x \wedge (x \equiv y))$
$\Rightarrow$"weakening, $\Rightarrow$-trans"
$(\forall y{:}\mathbb{B} \bullet P \sqsubseteq y \Rightarrow P \sqsubseteq x)$
$\equiv$"$\exists$-lub, y not in $P \sqsubseteq x$"
$(\exists y{:}\mathbb{B} \bullet P \sqsubseteq y) \Rightarrow P \sqsubseteq x$
$\equiv$"no-miracles"
$P \sqsubseteq x$ ❑

Theorem^4 $\Delta/\sqcap\mathbb{B}$: $\Delta(P \sqcap Q) \equiv \Delta P \wedge (P \equiv Q)$

Proof:

$\Delta(P \sqcap Q)$
$\equiv$"$\Delta/\sqcap$"
$(\Delta P \wedge P \sqsubseteq Q) \vee (\Delta Q \wedge Q \sqsubseteq P)$
$\equiv$"Δ-$\exists$ twice"
$((\exists x{:}\mathbb{B} \bullet P \equiv x) \wedge P \sqsubseteq Q) \vee ((\exists x{:}\mathbb{B} \bullet Q \equiv x) \wedge Q \sqsubseteq P)$
$\equiv$"$\wedge/\exists$ using $\Delta\sqsubseteq$"
$(\exists x{:}\mathbb{B} \bullet (P \equiv x) \wedge P \sqsubseteq Q) \vee (\exists x{:}\mathbb{B} \bullet (Q \equiv x) \wedge Q \sqsubseteq P)$
$\equiv$"$\wedge$-subst twice"
$(\exists x{:}\mathbb{B} \bullet (P \equiv x) \wedge x \sqsubseteq Q) \vee (\exists x{:}\mathbb{B} \bullet (Q \equiv x) \wedge x \sqsubseteq P)$
$\equiv$"$\mathbb{B}$-flat"
$(\exists x{:}\mathbb{B} \bullet (P \equiv x) \wedge (x \equiv Q)) \vee (\exists x{:}\mathbb{B} \bullet (Q \equiv x) \wedge (x \equiv P))$
$\equiv$"$\wedge$-subst twice"
$(\exists x{:}\mathbb{B} \bullet (P \equiv x) \wedge (P \equiv Q)) \vee (\exists x{:}\mathbb{B} \bullet (Q \equiv x) \wedge (Q \equiv P))$
$\equiv$"$\wedge/\exists$ twice using $\Delta\sqsubseteq$"
$((\exists x{:}\mathbb{B} \bullet P \equiv x) \wedge (P \equiv Q)) \vee ((\exists x{:}\mathbb{B} \bullet Q \equiv x) \wedge (Q \equiv P))$
$\equiv$"Δ-$\exists$ twice"
$(\Delta P \wedge (P \equiv Q)) \vee (\Delta Q \wedge (Q \equiv P))$
$\equiv$"$\wedge/\vee$ using $\equiv$-truth, $\wedge$-subst, $\wedge$-idem"
$\Delta P \wedge (P \equiv Q)$ ❑

Theorem^4 4-valued: $(P \equiv True) \vee (P \equiv False) \vee (P \equiv \bot_{\mathbb{B}}) \vee (P \equiv True \sqcap False)$

Proof:

$(P \equiv True) \vee (P \equiv False) \vee (P \equiv \bot_{\mathbb{B}}) \vee (P \equiv True \sqcap False)$
$\equiv$"$\sqsubseteq$True, $\sqsubseteq$False, $\equiv$True$\sqcap$False "
$\neg (P \sqsubseteq False) \vee \neg (P \sqsubseteq True) \vee (P \equiv \bot_{\mathbb{B}}) \vee (\tau P \wedge P \sqsubseteq True \wedge P \sqsubseteq False)$
$\equiv$"τ-defn, de Morgan"
True ❑

The following derived inference rule is available:

$$\text{truth cases}\quad \frac{(P\equiv True) \equiv (Q\equiv True),\ \ (P\equiv False) \equiv (Q\equiv False),\ \tau P\equiv\tau Q}{P \equiv Q}$$

It is justified by the following theorem, together with ∧-intro and equanimity:

Theorem^4 truth cases: $(P\equiv Q) \equiv ((P\equiv True) \equiv (Q\equiv True)) \wedge ((P\equiv False) \equiv (Q\equiv False)) \wedge (\tau P\equiv\tau Q)$
Proof: Similar to corresponding proof for the 3-valued case. ❑

The behaviour of the boolean operators with respect to $\bot_{\mathbb{B}}$ is governed by the following axioms:

Axiom^4 ¬⊥: $\neg\bot_{\mathbb{B}} \equiv \bot_{\mathbb{B}}$
Axiom^4 ∨⊥: $True\sqcap False \vee \bot_{\mathbb{B}} \equiv \bot_{\mathbb{B}}$

Theorem^4 τ¬: $\tau(\neg P) \equiv \tau P$
Proof: Apply τ-defn to left-hand side. ❑

Theorem^4 ⊑-exchange: $\neg P\sqsubseteq Q \equiv P\sqsubseteq\neg Q$
Proof: First show $\neg P\sqsubseteq True\equiv P\sqsubseteq False$ by applying ⊑True to left-hand side and appealing to axiom exchange. Similarly show $\neg P\sqsubseteq False \equiv P\sqsubseteq True$.

$\neg P\sqsubseteq Q$
≡"⊑-defn"
$(\tau(\neg P)\Rightarrow\tau Q) \wedge (\forall x,y{:}\mathbb{B} \bullet Q\sqsubseteq x \Rightarrow \neg P\sqsubseteq x)$
≡"τ¬ twice, 𝔹-instantiation"
$(\tau P\Rightarrow\tau(\neg Q)) \wedge (Q\sqsubseteq True \Rightarrow \neg P\sqsubseteq True) \wedge (Q\sqsubseteq False \Rightarrow \neg P\sqsubseteq False)$
≡"lemmas above"
$(\tau P\Rightarrow\tau(\neg Q)) \wedge (\neg Q\sqsubseteq False \Rightarrow P\sqsubseteq False) \wedge (\neg Q\sqsubseteq True \Rightarrow P\sqsubseteq True)$
≡"𝔹-instantiation and ⊑-defn"
$P\sqsubseteq\neg Q$ ❑

Theorem^4 ¬/⊓: $\neg(P\sqcap Q) \equiv \neg P\sqcap\neg Q$
Proof: Prove that each side refines the other using, in each case, theorems ⊑-exchange and ⊑-lub. ❑

Theorem^4 ∨/⊓: $P\vee(Q\sqcap R) \equiv (P\vee Q) \sqcap (P\vee R)$ and $(P\sqcap Q)\vee R \equiv (P\vee R) \sqcap (Q\vee R)$
Proof: Prove by cases. The result is trivially checked for the case P proper, and similarly for the case $Q\equiv R$. If $Q\not\equiv R$ then $Q\sqcap R$ is necessarily either ⊥ or True⊓False. If $Q\sqcap R$ is ⊥ then either Q or R is ⊥ and it is easy to check the equation for $P\in\{True\sqcap False, \bot\}$. If $Q\sqcap R$ is True⊓False then by symmetry between Q and R we need only check the equation for the cases $(Q\equiv True) \wedge (R\equiv False)$, $(Q\equiv True\sqcap False) \wedge (R\equiv False)$, and $(Q\equiv True\sqcap False) \wedge (R\equiv True)$ with $P\in\{True\sqcap False, \bot\}$. ❑

Theorem^4 ∧/⊓: $P\wedge(Q\sqcap R) \equiv (P\wedge Q) \sqcap (P\wedge R)$ and $(P\sqcap Q)\wedge R \equiv (P\wedge R) \sqcap (Q\wedge R)$

Theorem^4 ⇒/⊓: $P\Rightarrow(Q\sqcap R) \equiv (P\Rightarrow Q) \sqcap (P\Rightarrow R)$

Theorem^4 ¬⊑-mono: $P\sqsubseteq Q \Rightarrow \neg P\sqsubseteq\neg Q$

Theorem[4] ∨⊑-mono: $P \sqsubseteq Q \Rightarrow (P \vee R) \sqsubseteq (Q \vee R)$ and $Q \sqsubseteq R \Rightarrow (P \vee Q) \sqsubseteq (P \vee R)$

Theorem[4] ∧⊑-mono: $P \sqsubseteq Q \Rightarrow (P \wedge R) \sqsubseteq (Q \wedge R)$ and $Q \sqsubseteq R \Rightarrow (P \wedge Q) \sqsubseteq (P \wedge R)$

The behaviour of the quantifiers with respect to $\bot_{\mathbb{B}}$ is governed by the following axiom:

Axiom[4] τ/∀: $\tau(\forall x{:}T \bullet P) \equiv (\forall x{:}T \bullet \tau P) \vee (\exists x{:}T \bullet P \equiv \mathrm{False})$

Theorem[4] ∃⊑False: $(\exists x{:}T \bullet P) \sqsubseteq \mathrm{False} \equiv (\forall x{:}T \bullet P \sqsubseteq \mathrm{False})$

Proof:

$(\exists x{:}T \bullet P) \sqsubseteq \mathrm{False}$

≡"⊑False"

$(\exists x{:}T \bullet P) \not\equiv \mathrm{True})$

≡"∃-non-truth"

$(\forall x{:}T \bullet P \not\equiv \mathrm{True})$

≡"⊑False"

$(\forall x{:}T \bullet P \sqsubseteq \mathrm{False})$ ❑

Theorem[4] ∃⊑True: $(\exists x{:}T \bullet P) \sqsubseteq \mathrm{True} \equiv (\exists x{:}T \bullet P \sqsubseteq \mathrm{True})$

Theorem[4] τ/∃: $\tau(\exists x{:}T \bullet P) \equiv (\forall x{:}T \bullet \tau P) \vee (\exists x{:}T \bullet P \equiv \mathrm{True})$

Theorem[4] ∀⊑True: $(\forall x{:}T \bullet P) \sqsubseteq \mathrm{True} \equiv (\forall x{:}T \bullet P \sqsubseteq \mathrm{True})$

Theorem[4] ∀⊑False: $(\forall x{:}T \bullet P) \sqsubseteq \mathrm{False} \equiv (\exists x{:}T \bullet P \sqsubseteq \mathrm{False})$

Theorem[4] ∀/⊑: $(\forall x{:}T \bullet P \sqsubseteq Q) \Rightarrow ((\forall x{:}T \bullet P) \sqsubseteq (\forall x{:}T \bullet Q))$

Proof: Assume $(\forall x{:}T \bullet P \sqsubseteq Q)$.

$(\forall x{:}T \bullet P) \sqsubseteq (\forall x{:}T \bullet Q)$

≡"⊑-defn"

$(\tau(\forall x{:}T \bullet P) \Rightarrow \tau(\forall x{:}T \bullet Q)) \wedge (\forall y{:}\mathbb{B} \bullet (\forall x{:}T \bullet Q) \sqsubseteq y \Rightarrow (\forall x{:}T \bullet P) \sqsubseteq y)$

The second conjunct is routinely proved via 𝔹-instantiation. For the first conjunct:

$(\tau(\forall x{:}T \bullet P) \Rightarrow \tau(\forall x{:}T \bullet Q))$

≡"τ/∀ twice"

$(\forall x{:}T \bullet \tau P) \vee (\exists x{:}T \bullet P \equiv \mathrm{False}) \Rightarrow (\forall x{:}T \bullet \tau Q) \vee (\exists x{:}T \bullet Q \equiv \mathrm{False})$

⇐"∨-lub, and elementary properties of ⇒"

$((\forall x{:}T \bullet \tau P) \Rightarrow (\forall x{:}T \bullet \tau Q)) \wedge ((\exists x{:}T \bullet P \equiv \mathrm{False}) \Rightarrow (\exists x{:}T \bullet Q \equiv \mathrm{False}))$

⇐"∀/⇒"

$(\forall x{:}T \bullet \tau P \Rightarrow \tau Q) \wedge ((\exists x{:}T \bullet P \equiv \mathrm{False}) \Rightarrow (\exists x{:}T \bullet Q \equiv \mathrm{False}))$

≡"assumption, ⊑-defn"

$(\exists x{:}T \bullet P \equiv \mathrm{False}) \Rightarrow (\exists x{:}T \bullet Q \equiv \mathrm{False})$

≡"contrapositive using Δ∃ and ≡-truth"

$\neg(\exists x{:}T \bullet Q \equiv \mathrm{False}) \Rightarrow \neg(\exists x{:}T \bullet P \equiv \mathrm{False})$

≡"de Morgan twice"

$(\forall x{:}T \bullet Q \not\equiv False) \Rightarrow (\forall x{:}T \bullet P \not\equiv False)$
≡"⊑True twice"
$(\forall x{:}T \bullet Q \sqsubseteq True) \Rightarrow (\forall x{:}T \bullet P \sqsubseteq True)$
⇐"∀/⇒"
$(\forall x{:}T \bullet Q \sqsubseteq True \Rightarrow P \sqsubseteq True)$
≡"assumption, ⊑-defn"
True ❑

11. Equality

Equality (=) differs from equivalence in being strict with respect to null and ⊥, and in distributing over choice. Equality on flat types such as the booleans (and the integers were we to include them) can be axiomatised straightforwardly, but the axiomatisation presented below will be somewhat complicated to make it applicable to arbitrary types. To this end, we define equality in terms of the refinement relation on the type. This has the advantage that once the refinement relation is fixed for any type, an equality operator is immediately available.

To include **E** and **E3** in the treatment of equality, we need to augment them with the refinement relation. For that, we import ⊑ whole and entire from **EC**:

Axiom^{2,3} ⊑-defn: $E \sqsubseteq F \equiv (\forall x{:}T \bullet F \sqsubseteq x \Rightarrow E \sqsubseteq x)$ x fresh
Axiom^{2,3} ⊑-antisymm: $E \sqsubseteq F \wedge F \sqsubseteq E \Rightarrow E \equiv F$
Axiom^{2,3} no-miracles: $(\exists x{:}T \bullet E \sqsubseteq x)$ where E has type T and x fresh
Axiom^{2,3} 𝔹-flat: $(\forall x,y{:}\mathbb{B} \bullet x \sqsubseteq y \equiv (x \equiv y))$

For **E3** we also need to assert that ⊥ is the least element with respect to ⊑:

Axiom^{3} ⊥⊑: $\bot \sqsubseteq E$

⊑ obviously enjoys the same properties in **E** and **E3** as it does in **EC**. It will turn out that in **E**, equality and equivalence are identical on flat types.

Equality enjoys essentially the same properties in each logic in the family, but the proofs differ in small ways from logic to logic.

The axioms of equality (for arguments of type T) are as follows:

Axiom =-defn: $E{=}F \sqsubseteq True \equiv (\exists x{:}T|E \sqsubseteq x \bullet (\exists y{:}T|F \sqsubseteq y \bullet x \equiv y))$ x,y fresh
Axiom =-defn: $E{=}F \sqsubseteq False \equiv (\exists x{:}T|E \sqsubseteq x \bullet (\exists y{:}T|F \sqsubseteq y \bullet x \not\equiv y))$ x,y fresh

For **E4**, we need in addition

Axiom^{4} τ/=: $\tau(E{=}F) \equiv \tau E \wedge \tau F$

At the end of the section, we will comment further on the axiomatisation with regard to **E3** and **E4**.

The notion of "atomicity" of elements is important in the theory of equality. A term E is said to be "atomic", and satisfies ∇E, iff it denotes a proper element that has no refinements other than itself. All elements of flat types are trivially atomic. However, the logics do not exclude the possibility that some types may have two distinct proper elements where one refines the other; this is not uncommon in theories of functions. The defining axiom for atomicity is (where E has type T):

Axiom ∇-defn: $\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x)) \wedge \Delta E$ x fresh

For flat types, atomicity and properness coincide:

Theorem $\nabla\mathbb{B}$: $\nabla P \equiv \Delta P$

Proof:

∇P

$\equiv$"∇-defn"

$(\forall x{:}\mathbb{B} \bullet P \sqsubseteq x \Rightarrow (P \equiv x)) \wedge \Delta P$

$\equiv$"$\mathbb{B}$-flat, using ΔP in left-hand conjunct, $\wedge$-unit"

ΔP ❑

The second conjunct in axiom ∇-defn is redundant in **E**, **E3**, **EC**, and **E4**:

Theorem[2,3,c,4] ∇-defn: $\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x))$ x fresh

Proof:

$\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x))$

$\equiv$"∇-defn"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x)) \wedge \Delta E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x))$

$\equiv$"$\Rightarrow$-$\wedge$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x)) \Rightarrow \Delta E$

$\equiv$"Δ-$\exists$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x)) \Rightarrow (\exists x{:}T \bullet E \equiv x)$

$\equiv$"$\Rightarrow$-defn noting left-hand side proper, de Morgan"

$(\exists x{:}T \bullet \neg(E \sqsubseteq x \Rightarrow (E \equiv x))) \vee (\exists x{:}T \bullet E \equiv x)$

$\equiv$"$\exists/\vee$"

$(\exists x{:}T \bullet \neg(E \sqsubseteq x \Rightarrow (E \equiv x)) \vee (E \equiv x))$

$\equiv$"$\Rightarrow$-defn noting left-hand side proper, de Morgan"

$(\exists x{:}T \bullet (E \sqsubseteq x \wedge (E \not\equiv x)) \vee (E \equiv x))$

$\equiv$"strong $\equiv$, $\sqsubseteq$-refl, and elementary properties of the boolean operators"

$(\exists x{:}T \bullet E \sqsubseteq x)$ — no-miracles ❑

The definition of ∇ can also be simplified in **EB**:

Theorem[b] ∇-defn: $\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x)) \wedge (E \not\equiv \text{null})$ x fresh

Proof: If $E \equiv \text{null}$ then the equivalenve is trivially verified. If $E \not\equiv \text{null}$ then we must prove $\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (E \equiv x))$, and the proof of the preceding theorem does just this. ❑

Theorem =-symm: $E{=}F \equiv F{=}E$

Proof: Follows from the symmetry in the defining axioms for =. ❑

Theorem[c,b,4] =/⊓: $E{=}(F \sqcap G) \equiv E{=}F \sqcap E{=}G$

Proof: Using ⊑-defn:

(a) $E{=}(F \sqcap G) \sqsubseteq \text{True}$

$\equiv$"=-defn"

$(\exists x{:}T | E \sqsubseteq x \bullet (\exists y{:}T | F \sqcap G \sqsubseteq y \bullet x \equiv y))$

$\equiv$"trading noting Δ⊑"

$(\exists x{:}T \bullet E \sqsubseteq x \wedge (\exists y{:}T \bullet F \sqcap G \sqsubseteq y \wedge (x \equiv y)))$

$\equiv$"∧/∃ noting y not in E⊑x"

$(\exists x{:}T \bullet (\exists y{:}T \bullet E \sqsubseteq x \wedge F \sqcap G \sqsubseteq y \wedge (x \equiv y)))$

$\equiv$"⊓-defn"

$(\exists x{:}T \bullet (\exists y{:}T \bullet E \sqsubseteq x \wedge (F \sqsubseteq y \vee G \sqsubseteq y) \wedge (x \equiv y)))$

$\equiv$"∧/∨ noting all terms proper, and ∃/∨"

$(\exists x{:}T \bullet (\exists y{:}T \bullet E \sqsubseteq x \wedge F \sqsubseteq y \wedge (x \equiv y)) \vee (\exists x{:}T \bullet (\exists y{:}T \bullet E \sqsubseteq x \wedge G \sqsubseteq y \wedge (x \equiv y)))$

$\equiv$"trading and distributing as above"

$(\exists x{:}T | E \sqsubseteq x \bullet (\exists y{:}T | F \sqsubseteq y \bullet x \equiv y)) \vee (\exists x{:}T | E \sqsubseteq x \bullet (\exists y{:}T | G \sqsubseteq y \bullet (x \equiv y))$

$\equiv$"=-defn"

$E{=}F \sqsubseteq \text{True} \vee E{=}G \sqsubseteq \text{True}$

$\equiv$"⊓-defn"

$E{=}F \sqcap E{=}G \sqsubseteq \text{True}$

(b) $E{=}(F \sqcap G) \sqsubseteq \text{False} \equiv E{=}F \sqcap E{=}G \sqsubseteq \text{False}$ is proved similarly.

(c) The following is applicable to **E4** only:

$\tau(E{=}F \sqcap E{=}G)$

$\equiv$"τ/⊓"

$\tau(E{=}F) \wedge \tau(E{=}G)$

$\equiv$"τ/="

$\tau E \wedge \tau F \wedge \tau E \wedge \tau G$

$\equiv$"∧-idem"

$\tau E \wedge \tau F \wedge \tau G$

$\equiv$"τ/⊓ and τ/="

$\tau(E{=}(F \sqcap G))$ ❑

Theorem[b] =null: $E{=}\text{null} \equiv \text{null}$

Proof: By Theorem is-null we need only show $\neg(\exists x{:}\mathbb{B} \bullet E{=}\text{null} \sqsubseteq x)$, and this follows routinely from 𝔹-instantiation and the defining axioms for =. ❑

Theorem[3,4] =⊥: $E{=}\bot \equiv \bot$

Proof: For **E4**:

$E{=}\bot \equiv \bot$

$\equiv$"τ-defn"
$\neg\tau(E{=}\bot)$
$\equiv$"τ/="
$\neg(\tau E \wedge \tau\bot)$
$\equiv$"¬τ⊥"
True

For **E3**, $E{=}\bot \equiv \bot$ follows from $E{=}\bot\not\equiv \text{False} \wedge E{=}\bot\not\equiv\text{True}$ by Theorem 3-valued.

(a) $E{=}\bot\not\equiv\text{False}$
$\equiv$"⊑True"
$E{=}\bot\sqsubseteq\text{True}$
$\equiv$"=-defn"
$(\exists x{:}T|E\sqsubseteq x \bullet (\exists y{:}T|\bot\sqsubseteq y \bullet x\equiv y))$
$\equiv$"trading noting Δ⊑"
$(\exists x{:}T \bullet E\sqsubseteq x \wedge (\exists y{:}T \bullet \bot\sqsubseteq y \wedge (x\equiv y)))$
$\equiv$"⊥⊑"
$(\exists x{:}T \bullet E\sqsubseteq x \wedge (\exists y{:}T \bullet \text{True} \wedge x\equiv y))$
$\equiv$"trading, one-point, habitation, and elementary properties of ∧"
$(\exists x{:}T \bullet E\sqsubseteq x)$
$\equiv$"no-miracles"
True

(b) $E{=}\bot\not\equiv\text{True}$
$\equiv$"⊑False"
$E{=}\bot\sqsubseteq \text{False}$
$\equiv$"=-defn"
$(\exists x{:}T|E\sqsubseteq x \bullet (\exists y{:}T|\bot\sqsubseteq y \bullet x\not\equiv y))$
$\equiv$"⊥⊑"
$(\exists x{:}T \bullet E\sqsubseteq x \wedge (\exists y{:}T \bullet x\not\equiv y))$
$\equiv$"Theorem no-units below"
$(\exists x{:}T \bullet E\sqsubseteq x)$
$\equiv$"no-miracles"
True ❑

The proof of the preceding theorem in **E3** makes use of Theorem no-units which states, in effect, that each type has at least two elements (we already know from theorem habitation that each type has at least one element). Therefore we postulate axiom no-units below, and and give an alternatice formulation of it in Theorem no-units. We will show at the end of the section how axiom no-units can be dispensed with.

Axion[3] no-units: $(\exists x{:}T \bullet (\exists y{:}T \bullet x\not\equiv y))$

Theorem[3] no-units: $(\forall x{:}T \bullet (\exists y{:}T \bullet x\not\equiv y))$

Proof:

$(\forall x{:}T \bullet (\exists y{:}T \bullet x \not\equiv y))$

$\equiv$"axiom no-units"

$(\exists x{:}T \bullet (\exists y{:}T \bullet x \not\equiv y)) \Rightarrow (\forall x{:}T \bullet (\exists y{:}T \bullet x \not\equiv y))$

$\equiv$"$\Rightarrow/\forall$"

$(\forall x{:}T \bullet (\exists x{:}T \bullet (\exists y{:}T \bullet x \not\equiv y)) \Rightarrow (\exists y{:}T \bullet x \not\equiv y))$

$\equiv$"interchange"

$(\forall x{:}T \bullet (\exists y{:}T \bullet (\exists x{:}T \bullet x \not\equiv y)) \Rightarrow (\exists y{:}T \bullet x \not\equiv y))$

$\equiv$"$\exists$-lub"

$(\forall x{:}T \bullet (\forall y{:}T \bullet (\exists x{:}T \bullet x \not\equiv y) \Rightarrow (\exists y{:}T \bullet x \not\equiv y)))$

$\equiv$"lemma below"

$(\forall x{:}T \bullet (\forall y{:}T \bullet \text{True}))$ — True generalised twice

It remains to prove that $(\exists x{:}T \bullet x \not\equiv y) \Rightarrow (\exists y{:}T \bullet x \not\equiv y)$ holds for arbitrary x and y. Renaming dummies, we have to show $(\exists u{:}T \bullet u \not\equiv y) \Rightarrow (\exists v{:}T \bullet x \not\equiv v)$. We consider the two cases $x \equiv y$ and $x \not\equiv y$. If $x \equiv y$ the result is trivial. If $x \not\equiv y$ then the truth of both antecedent and consequent follow from instantiation (in the form $P \Rightarrow (\exists x{:}T \bullet P)$ where P is replaced by $x \not\equiv y$). ❑

Theorem =-truth: $(E{=}F \equiv \text{True}) \equiv \nabla E \wedge (E \equiv F)$

Proof: We first carry out the proof in **E4**. If $\neg\tau E$ or $\neg\tau F$ then the equivalence is easily shown. Assume τE and τF.

$E{=}F \equiv \text{True}$

$\equiv$"$\sqsubseteq$False"

$\neg(E{=}F \sqsubseteq \text{False})$

$\equiv$"=-defn"

$\neg(\exists x{:}T|E \sqsubseteq x \bullet (\exists y{:}T|F \sqsubseteq y \bullet x \not\equiv y))$

$\equiv$"de Morgan"

$(\forall x{:}T|E \sqsubseteq x \bullet (\forall y{:}T|F \sqsubseteq y \bullet x \equiv y))$

$\equiv$"trading"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"$\Rightarrow/\wedge$, and elementary properties of $\Rightarrow$ and $\wedge$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow F \sqsubseteq y \wedge (x \equiv y)))$

$\equiv$"$\wedge$-subst"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow F \sqsubseteq x \wedge (x \equiv y)))$

$\equiv$"$\Rightarrow/\wedge$ and $\forall/\wedge$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow F \sqsubseteq x) \wedge (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"$\exists$-lub using y not free in $F \sqsubseteq x$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow ((\exists y{:}T \bullet F \sqsubseteq y) \Rightarrow F \sqsubseteq x) \wedge (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"no-miracles"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow F \sqsubseteq x \wedge (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"$\Rightarrow/\wedge$ and $\forall/\wedge$"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow F \sqsubseteq x) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"assuming τE and τF"

$(\tau F \Rightarrow \tau E) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow F \sqsubseteq x) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

$\equiv$"$\sqsubseteq$-defn"

$(F \sqsubseteq E) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

≡"by symmetry of = we also infer $F \sqsubseteq E$, ⊑-antisymm"

$(E \equiv F) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet F \sqsubseteq y \Rightarrow (x \equiv y)))$

≡"∧-subst"

$(E \equiv F) \wedge (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet E \sqsubseteq y \Rightarrow (x \equiv y)))$

≡"lemma below"

$(E \equiv F) \wedge \nabla E$

It remains to prove the following lemma:

If τE then $\nabla E \equiv (\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet E \sqsubseteq y \Rightarrow (x \equiv y)))$

We can prove this by bi-implication as both sides are proper. The proof from left to right follows easily from the defining properties of ∇ which allows us to replace each ⊑ with ≡. Going from right to left:

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet E \sqsubseteq y \Rightarrow (x \equiv y)))$

⇒"⊑-refl, and monotonicity properties of ⇒"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\forall y{:}T \bullet E \sqsubseteq y \Rightarrow (x \sqsubseteq y)))$

≡"τE given, τx"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (\tau x \Rightarrow \tau E) \wedge (\forall y{:}T \bullet E \sqsubseteq y \Rightarrow (x \sqsubseteq y)))$

≡"⊑-defn"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow x \sqsubseteq E)$

≡"⊑-antisymm and elementary properties of ⇒"

$(\forall x{:}T \bullet E \sqsubseteq x \Rightarrow (x \equiv E))$

≡"∇-defn"

∇E

The proofs in **E**, **E3**, and **EC** are a minor simplication of that given above, the simplification arising from the absence of any need to consider τ. The proof in **EB** proceeds almost identically to that given above, with $E \not\equiv \text{null}$ taking over the role of τE, and analogously for $F \not\equiv \text{null}$. ❑

In E, equality coincides with equivalence on flat types:

Theorem[2] =-truth: $E{=}F \equiv (E \equiv F)$ if ∇E

Proof: $E{=}F$

≡"all proper"

$E{=}F \equiv \text{True}$

≡"=-truth"

$\nabla E \wedge (E \equiv F)$

≡"∇E"

$E \equiv F$ ❑

Theorem =-≡: $E{=}F \Rightarrow (E \equiv F)$

Proof: Follows immediately from ⇒-defn and =-truth. ❑

Equality is reflexive only for atomic elements:

Theorem =-refl: ∇E ⇒ E=E

Proof: Using True-elim:

(∇E ⇒ E=E) ≡ True

≡"⇒-truth"

∇E ⇒ (E=E ≡ True)

≡"=-truth"

∇E ⇒ ∇E ∧ (E≡E)

≡"≡-refl, ∧-unit, ⇒-refl"

True ❑

Theorem =-trans: E=F ∧ F=G ⇒ E=G

Proof: Using True-elim and ⇒-truth:

(E=F ∧ F=G) ≡ True

≡"∧-truth"

(E=F≡True) ∧ (F=G≡True)

≡"=-truth"

∇E ∧ (E≡F) ∧ ∇F ∧ (F≡G)

⇒"≡-trans and weakening"

∇E ∧ (E≡G)

≡"=-truth"

E=G ≡ True ❑

Theorem =⊑-mono: E⊑F ⇒ (E=G)⊑(F=G) and F⊑G ⇒ (E=F)⊑(E=G)

Proof: For **EC**, **EB**, and **E4**, the proof follows readily from ⊑⊓ and =/⊓. For **E** and **E3**, use ⊑-defn. ❑

Theorem =-𝔹: P=Q ≡ (P∧Q) ∨ (¬P∧¬Q)

Proof: The proof in **E**, **E3**, and **EC** consists of steps (a) and (b) below; **E4** requires in addition step (c).

(a) (P∧Q) ∨ (¬P∧¬Q) ≡ True

≡"∨-truth, ∧-truth"

((P≡True) ∧ (Q≡True)) ∨ ((¬P≡True) ∧ (¬Q≡True))

≡"∧-subst twice"

((P≡Q) ∧ (Q≡True)) ∨ ((¬P≡¬Q) ∧ (¬Q≡True))

≡"≡-mirror, ∧/∨"

(P≡Q) ∧ ((Q≡True) ∨ (¬Q≡True))

≡"exchange, boolean properness"

(P≡Q) ∧ ΔQ

≡"∇𝔹"

(P≡Q) ∧ ∇Q

≡"=-truth"

P=Q ≡ True

(b) P=Q ≡ False

≡"⊑True"

$\neg(P{=}Q \sqsubseteq \text{True})$

$\equiv$"=-defn"

$\neg(\exists x{:}\mathbb{B}|P\sqsubseteq x \bullet (\exists y{:}\mathbb{B}|Q\sqsubseteq y \bullet x\equiv y))$

$\equiv$"de Morgan"

$(\forall x{:}\mathbb{B}|P\sqsubseteq x \bullet (\forall y{:}\mathbb{B}|Q\sqsubseteq y \bullet x\not\equiv y))$

$\equiv$"de Morgan"

$(\forall x{:}\mathbb{B}|P\sqsubseteq x \bullet (\forall y{:}\mathbb{B}|Q\sqsubseteq y \bullet x\not\equiv y))$

$\equiv$"trading, $\Rightarrow/\forall$ noting y not in $P\sqsubseteq x$, shunting"

$(\forall x{:}\mathbb{B} \bullet (\forall y{:}\mathbb{B} \bullet P\sqsubseteq x \wedge Q\sqsubseteq y \Rightarrow (x\not\equiv y)))$

$\equiv$"$\mathbb{B}$-instantiation"

$(P\sqsubseteq\text{True} \wedge Q\sqsubseteq\text{True} \Rightarrow (\text{True}\not\equiv\text{True})) \wedge (P\sqsubseteq\text{True} \wedge Q\sqsubseteq\text{False} \Rightarrow (\text{True}\not\equiv\text{False})) \wedge$

$(P\sqsubseteq\text{False} \wedge Q\sqsubseteq\text{True} \Rightarrow (\text{False}\not\equiv\text{True})) \wedge (P\sqsubseteq\text{False} \wedge Q\sqsubseteq\text{False} \Rightarrow (\text{False}\not\equiv\text{False}))$

$\equiv$"$\sqsubseteq$True, $\sqsubseteq$False, two values, $\equiv$-refl, $\wedge$-unit"

$(P\not\equiv\text{False} \wedge Q\not\equiv\text{False} \Rightarrow \text{False}) \wedge (P\not\equiv\text{True} \wedge Q\not\equiv\text{true} \Rightarrow \text{False})$

$\equiv$"$\Rightarrow$-defn, $\vee$-unit, and elementary properties of $\equiv$"

$((P\equiv\text{False}) \vee (Q\equiv\text{False})) \wedge ((P\equiv\text{true}) \vee (Q\equiv\text{True}))$

$\equiv$"exchange"

$((P\equiv\text{False}) \vee (Q\equiv\text{False})) \wedge ((\neg P\equiv\text{False}) \vee (\neg Q\equiv\text{False}))$

$\equiv$"$\vee$-Falsity and $\wedge$-Falsity"

$(P\wedge Q) \vee (\neg P\wedge\neg Q) \equiv \text{False}$

(c) For **E4**, it remains to show $\tau(P{=}Q) \equiv \tau((P\wedge Q) \vee (\neg P\wedge\neg Q))$. If $P{=}Q$ is $\perp$ then either P or Q is $\perp$ and in either case we easily infer that $(P\wedge Q) \vee (\neg P\wedge\neg Q)$ is $\perp$. If $P{=}Q$ is not $\perp$ then neither P nor Q is $\perp$ and so they are both drawn from {True, False, True$\sqcap$False}. Clearly, in no case can $(P\wedge Q) \vee (\neg P\wedge\neg Q)$ yield $\perp$.

For the proof in EB, consider first the case when P or Q is null; in this case, the equivalence is easily verified. Otherwise, observe that neither side is null (the argument is similar to that in step (c) above with $\perp$ replaced with null). We now apply $\sqsubseteq$-defn, and aim to show $P{=}Q\sqsubseteq x \equiv (P\wedge Q)\vee(\neg P\wedge\neg Q)\sqsubseteq x$ where x is instantiated in turn with True and False. Under the non-null assumption, this is just $(P{=}Q\not\equiv x) \equiv ((P\wedge Q)\vee(\neg P\wedge\neg Q)\not\equiv x)$ by Theorems $\sqsubseteq$True and $\sqsubseteq$False of **EB**. From this point on, the proof is just steps (a) and (b) above. ❑

To prove that equality is strict with respect to $\perp$ in **E3**, we postulated axiom no-units. We can also derive no-units within **E4**. This is not too serious a restriction, and in fact we imposed it only so that we could axiomatise equality uniformly for all the logics. We can dispense with it by reformulating the second defining axiom for equality within **E3** and **E4** as:

Axiom^{3,4} =-defn: $E{=}F\sqsubseteq\text{False} \equiv (E\equiv\perp) \vee (F\equiv\perp) \vee (\exists x{:}T|E\sqsubseteq x \bullet (\exists y{:}T|F\sqsubseteq y \bullet x\not\equiv y))$ x,y fresh

We can express this alternatively in **E3** and **E4**, respectively, as:

Axiom^{3} =-defn: $E{=}F\sqsubseteq\text{False} \equiv \neg\Delta E \vee \neg\Delta F \vee (\exists x{:}T|E\sqsubseteq x \bullet (\exists y{:}T|F\sqsubseteq y \bullet x\not\equiv y))$ x,y fresh

Axiom^{4} =-defn: $E{=}F\sqsubseteq\text{False} \equiv \neg\tau E \vee \neg\tau F \vee (\exists x{:}T|E\sqsubseteq x \bullet (\exists y{:}T|F\sqsubseteq y \bullet x\not\equiv y))$ x,y fresh

Proofs in **E3** and **E4** using the above axioms differ only in only small ways from those given above.

References

[1] E. W. Dijkstra and C. S. Scholten, *Predicate Calculus and Program Semantics* (Springer-Verlag, New York, 1990).

[2] D. Gries and F. B. Schneider, *A Logical Approach to Discrete Math* (Springer-Verlag, New York, 1993).

[3] H. Barringer, J. H. Cheng, and C.B. Jones, A logic covering undefinedness in program proofs, *Acta Informatica* 21 (1984) 251-269.

[4] C. B. Jones and C. A. Middelburg, A typed logic of partial functions reconstructed classically, *Acta Informatica* 31 (1994) 399-430.

[5] J. M. Morris and A. Bunkenburg, E3: A Logic for Reasoning Equationally in the Presence of Partiality, *Science of Computer Programming* 34 (1998) 141-158.

[6] J. M. Morris and A. Bunkenburg, Partiality and Nondeterminacy in Program Proofs, *Formal Aspects of Computing* 10 (1999) 76-96.

[7] A. Bunkenburg and J. M. Morris, A Theory of Bunches, *Acta Informatica*, to appear, 1999.

[8] A. G. Hamilton, *Logic for Mathematicians* (Cambridge University Press, Cambridge, 1988).