Linear Programming (L.P.)

Problems of allocation of scarce resources in "best" possible manner.

- **Max profit**
- **Min cost**

Conditions of L.P. problem:-

1. **Decision variables $\geq$ 0**
2. **"Best" criterion described by linear function - Objective Function**
3. **Rules (e.g. scarcity of resources) described by linear functions - Constraints.**

Advantages

- **Efficient solution methods available**
- **Data variation easily handed**
- **Often possible to approximate non-linearities.**

Extensions

Non-linear programming

- **Quadratic programming**
- **Integer programming**
- **Dynamic programming**

Areas of Application

- **Petro-chemical industries (50% of all L.P. applications)**
- **Agri-food industries**
  - farm planning
  - feed mix
  - product mix
- **Distribution**
  - warehouse location
  - minimising transport costs
- **Public Services**
  - water supply
  - roads

Formulation of Linear Programming Problems

1. **Identify variables**
2. **Identify restrictions**
3. **Identify criterion**

Examples

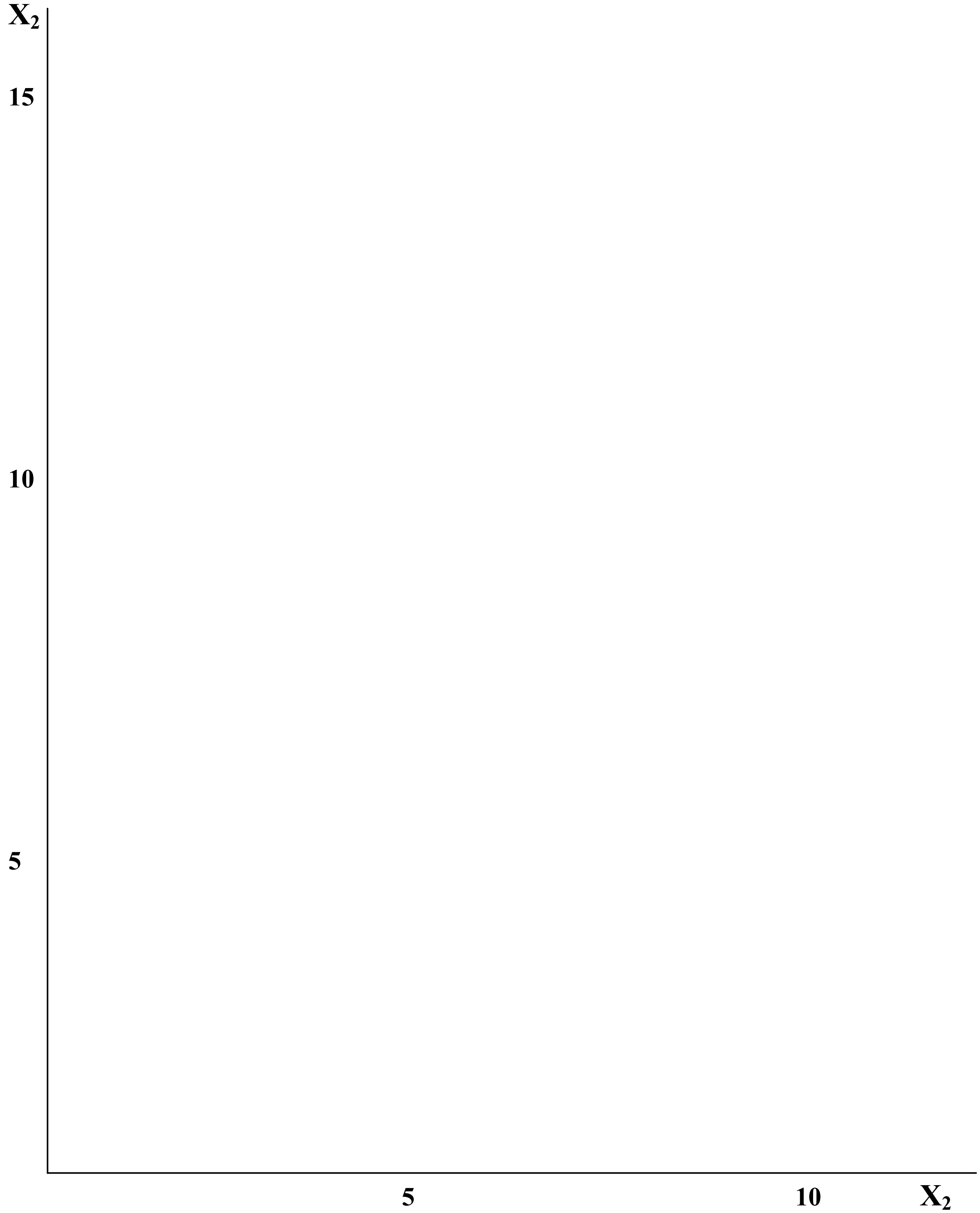
1. Product mix:

The Handy-Dandy Company makes three types of kitchen appliances (*A, B* and *C*). To make each of these appliance types, just two inputs are required - labour and materials. Each unit of *A* made requires 7 hours of labour and four Kg of materials; for each unit of *B* made the requirements are 3 hours of labour and 4 Kg of materials, while for *C* the unit requirements are 6 hours of labour and 5 Kg of material. The company expects to make a profit of €40 for every unit of *A* sold, and the profit per unit for *B* and *C* are €20 and €30 respectively. Given that the company has available to it 150 hours of labour and 200 Kg of material each day, formulate this as a linear programming problem.

Inspection problem

A company employs two grades of quality control inspector to examine pieces being produced on a production line. A grade one inspector can inspect at the rate of 25 pieces per hour, with 98% accuracy and for this they are paid €16 per hour. Grade two inspectors inspect pieces at the slower rate of 15 per hour and with 95% accuracy, and they are paid €12 per hour. The company can call on up to eight grade one inspectors and up to 10 grade two inspectors, but inspectors who are not called upon do not have to be paid. Any errors which are made in the inspection process cost the company €8 each. The company requires that at least 1800 pieces must be inspected each day (8 hours). Formulate this as a linear programming problem.

$$\begin{aligned}
&\textit{Max } 10X_1 + 9X_2 \\
&\textit{S.T.}\quad X_1 \le 8 \\
&\qquad\;\; X_2 \le 10 \\
&\qquad\;\; 5X_1 + 3X_2 \le 45 \\
&\qquad\;\; X_1,\ X_2 \ge 0
\end{aligned}$$



Alternative/Multiple Optimal Solutions

e.g.

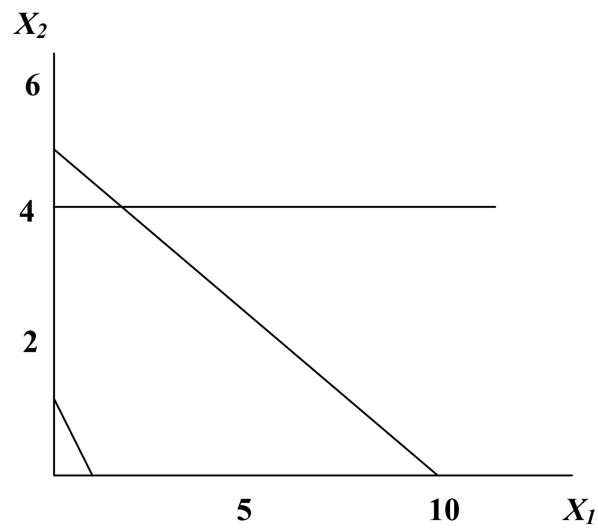
Max $Z = X_1 + 2X_2$

s.t. $2X_1 + 4X_2 \leq 20$

$X_1 + X_2 \geq 1$

$X_2 \leq 4$

$X_1, X_2 \geq 0$



Unbounded Solution

Possible to find further feasible solutions while continuously improving the objective function value.

e.g.

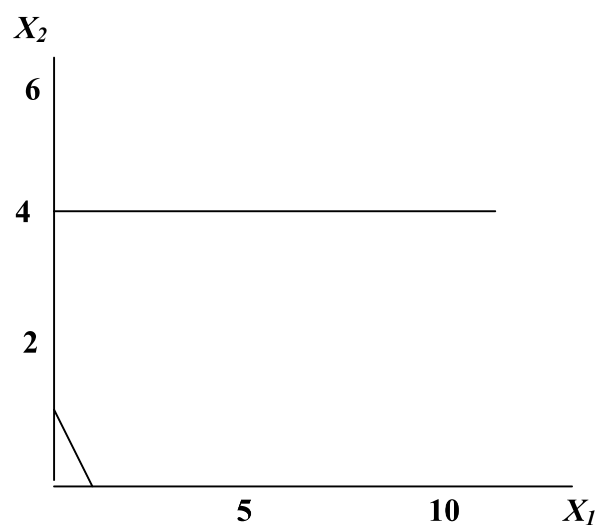
Max $Z = X_1 + 2X_2$

s.t.

$X_1 + X_2 \geq 1$

$X_2 \leq 4$

$X_1, X_2 \geq 0$



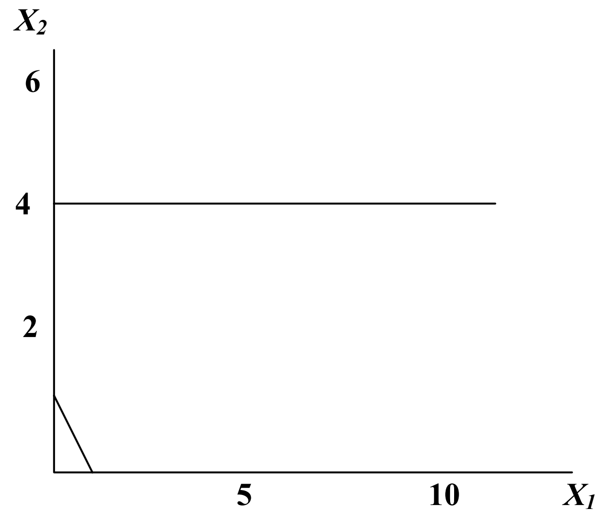
Infeasible problem

No feasible solution
Feasible region = {∅}

e.g.
Max $Z = X_1 + 2X_2$

s.t.

$$X_1 + X_2 \leq 1$$
$$X_2 \geq 4$$
$$X_1, X_2 \geq 0$$



Note: Unique optimum always occurs at a corner of the feasible region - fundamental property of linear programming problems.

- **basis for Simplex Method for solving L.P.'s developed by G. B. Dantzig (1947).**

Standard Form of Linear Programming Problem

Representation of problem with *m* constraints and *n* variables:

Maximise (or Minimise)

$$Z = c_1X_1 + c_2X_2 + \ldots\ldots\ldots\ldots + c_nX_n$$

subject to

$$\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \ldots\ldots\ldots\ldots + a_{1n}X_n &= b_1 \\
a_{21}X_1 + a_{22}X_2 + \ldots\ldots\ldots\ldots + a_{2n}X_n &= b_2 \\
\ldots \qquad\qquad\qquad \ldots\ldots \qquad \ldots\ldots \quad &\ \ \ldots \\
\ldots \qquad\qquad\qquad \ldots\ldots \qquad \ldots\ldots \quad &\ \ \ldots \\
a_{m1}X_1 + a_{m2}X_2 + \ldots\ldots\ldots\ldots + a_{mn}X_n &= b_m
\end{aligned}$$

$$X_1, X_2, X_3, \ldots\ldots\ldots\ldots X_n \geq 0$$
$$b_1, b_2, b_3, \ldots\ldots\ldots\ldots b_m \geq 0$$

Note:

1. **Objective function is maximise or minimise.**
2. **Constraints are expressed as equations**
3. **All variables are ≥ 0.**
4. **All right-hand-side (RHS) constants are ≥ 0.**

Vector-matrix notation:

Max (Min) $Z = \underline{c}\,\underline{x}$
subject to $\underline{A}\,\underline{x} = \underline{b}$
$\underline{x} \geq \underline{0}$
$\underline{b} \geq \underline{0}$

$\underline{A}$ **(m x n) coefficient matrix**
$\underline{x}$ **(n x 1) decision vector**
$\underline{b}$ **(m x 1) requirement vector**
$\underline{c}$ **(1 x n) cost(profit) vector**

<u>Handling Inequality Constraints</u>

Inequalities need to be converted into equations using slack or surplus variables.

e.g. $X_1 \leq 8$ **becomes** $X_1 + X_3 = 8$
$X_3 \geq 0$
X_3 - **<u>Slack variable</u>**

$200X_1 + 120X_2 \geq 1800$ **becomes**
$200X_1 + 120X_2 - X_4 = 1800$
$X_4 \geq 0$
X_4 - **<u>Surplus variable</u>**

Handling Variables with Unrestricted Sign

e.g. Investor has €100 cash and can borrow to invest.

X_1 = amount invested

Investment constraint:-

$$X_1 + X_2 = 100$$

where X_2 can be positive or negative.

Replace X_2 by $(X_2' - X_2'')$

$$X_2' \geq 0, \quad X_2'' \geq 0$$

At solution either $X_2' > X_2''$

or $X_2' < X_2''$

Linear Equations

$$X_1 - 2X_2 + X_3 - 4X_4 = 2 \quad (1) \quad \} \ldots\ldots S_1$$
$$X_1 - X_2 - X_3 - 3X_4 = 4 \quad (2) \quad \}$$

> 1 solution - Solution set

Elementary row operations (pivoting)

- **Multiply an equation by a number**
- **Add to any equation a constant multiple of any other equation**

e.g. multiply (1) by -1 and add it to (2) →

$$X_1 - 2X_2 + X_3 - 4X_4 = 2 \quad (3) \quad \} \ldots\ldots S_2$$
$$X_2 - 2X_3 + X_4 = 2 \quad (4) \quad \}$$

Multiply (4) by 2 and add to (3) →

$$X_1 - 3X_3 - 2X_4 = 6 \quad (5) \quad \} \ldots\ldots S_3$$
$$X_2 - 2X_3 + X_4 = 2 \quad (6) \quad \}$$

S_3 has same soltuion set as S_1.
Possible solution:- $X_1 = 6$ $X_2 = 2$ $X_3 = 0$ $X_4 = 0$
(basic solution)

X_1, X_2 - **Basic variables**
X_3, X_4 - **Non-basic variables**
S_3 - **Canonical system**

Simplex Method

- Requires initial basic feasible solution (B.F.S.)
- Improve by finding another B.F.S. with better objective function value
- Continue until no improvement possible

Max $Z = 3X_1 + 2X_2$

s.t. $2X_1 + X_2 \leq 6$

$X_1 + 2X_2 \leq 8$

$X_1, X_2 \geq 0$

Standard form:-

Max $Z = 3X_1 + 2X_2$

s.t. $2X_1 + X_2 + S_1 = 6$

$X_1 + 2X_2 + S_2 = 8$

$X_1, X_2, S_1, S_2 \geq 0$

S_1, S_2 - **slack variables**

B.F.S. $S_1 = 6 \quad S_2 = 8 \quad X_1 = 0 \quad X_2 = 0$

S_1, S_2 - **basic variables**

Simplex tableau:-

	X_1	X_2	S_1	S_2	b
Z					

- **find another B.F.S. with higher Z value**
- **scan Z row for negative entries (maximisation)**
- **choose largest (absolute)**

Make X_1 basic

Maximum value of X_1 - <u>Minimum ratio test</u>

$$\frac{\text{b value}}{\text{coefficient}} = \text{Min}\left(6/2,\ 8/1\right) = 6/2$$

Denominators > 0

X_1 replaces S_1 as a basic variable

To make X_1 basic, X_1 column:- 1 in 1st equation
0 elsewhere

<u>Pivots</u>

- **Multiply row S_1 row by 1/2 → PIVOT ROW**
- **Add 3*times pivot row to Z row → new Z row**
- **S_2 row minus pivot row → new S_2 row**

New tableau:-

	X_1	X_2	S_1	S_2	b
Z					

Improvement possible?

Negative entry in Z row → yes. Make X_2 basic.
Minimum ratio test: (3 / (1/2) , 5 / (3/2))
So S_2 leaves basis.

Pivots

- **Multiply S_2 by (2/3) → Pivot row**
- **X_1 row minus (1/2)* pivot row → new X_1 row**
- **Z row + (1/2)*pivot row → new Z row.**

New tableau:-

	X_1	X_2	S_1	S_2	b
Z					

No negatives in Z row, so solution is optimal.
$X_1^* = 4/3$; $X_2^* = 10/3$; $Z^* = 32/3$

Simplex Algorithm

(Maximistion with ≤ constraints)

Step 0: Use standard from to obtain initial B.F.S.

Step 1: Select variable to enter basis (largest negative entry in z-row)
If none, STOP

Step 2: Select variable to leave basis (minimum ratio test)

Step 3: Calculate new B.F.S. (pivot operations)

Go to step 1.

Minimisation:- Modify step 1 → Select variable with largest positive entry.

Computational Problems

- **Ties in selection of non-basic variable to enter basis:**
  - **arbitrary choice**
  - **minimum subscript rule**

- **Ties in ratio rule:**

  → <u>Degenerate</u> solution – one or more basic variable takes on a value of zero.

Example:

	X_1	X_2	X_3	X_4	X_5	X_6	b
Z	0	0	0	-2	0	-1	4
X_1	1	0	0	1	-1	0	2
X_2	0	1	0	2	0	1	4
X_3	0	0	1	1	1	1	3

Choosing *X_1* as leaving variable:-

	X_1	X_2	X_3	X_4	X_5	X_6	b
Z	2	0	0	0	-2	-1	8
X_4	1	0	0	1	-1	0	2
X_2	-2	1	0	0	2	1	0
X_3	-1	0	1	0	2	1	1

Next iteration X_5 leaves basis and X_2 leaves, but value of objective function does not improve.

Normally degeneracy rectifies itself. Re-appearance of a previous b.f.s. – Cycling (rare in practice).

- **If all coefficients in column of entering variable are negative or zero (i.e. minimum ratio test fails) → solution is unbounded.**

Example (maximisation):

	X_1	X_2	S_1	S_2	b
Z	-2	-3	0	0	0
S_1	1	-1	1	0	10
S_2	2	0	0	1	40

• ***Zero*** **entry for non-basic variable in Z row →** <u>**alternative solution**</u> **exists.**

Example (maximisation):

	X_1	X_2	S_1	S_2	S_3	b
Z	0	0	0	1	0	14
S_1	0	0	1	-1/5	8/5	6
X_2	0	1	0	1/5	-3/5	1
X_1	1	0	0	1/5	2/5	4

Could make S_3 basic without changing value of Z → alternative optimum.
Alternative solution $X_1 = 5/2 \quad X_2 = 13/4$
$S_1 = 0 \quad S_2 = 0 \quad S_3 = 15/4$

• **If negative appears in b column, problem is** <u>**infeasible**</u>**.**

Artificial Variables

- **Needed when there is no obvious starting solution.**

- **If there is no basic variable for an equation, one is 'invented' – Artificial variable.**

- **Values of artificial variables are driven to zero by the simplex method.**

Example:

Min $Z = 4X_1 + X_2$

s.t.

$$3X_1 + X_2 = 3$$
$$4X_1 + 3X_2 \geq 6$$
$$X_1 + 2X_2 \leq 4$$
$$X_1, X_2 \geq 0$$

Standard form:-

Min $Z = 4X_1 + X_2$

s.t.

$$3X_1 + X_2 = 3$$
$$4X_1 + 3X_2 - S_1 = 6$$
$$X_1 + 2X_2 + S_2 = 4$$
$$X_1, X_2, S_1, S_2 \geq 0$$

Not in canonical form (no basic variables for first and second equations). Augment these equations with artificial variables R_1 and R_2, as follows:

$$3X_1 + X_2 + R_1 = 3$$
$$4X_1 + 3X_2 - S_1 + R_2 = 6$$

To drive values of R_1 and R_2 to zero (eliminate from basis) they are penalised in the objective function.

Min $Z = 4X_1 + X_2 + MR_1 + MR_2$

s.t.

$$3X_1 + X_2 + R_1 = 3$$
$$4X_1 + 3X_2 - S_1 + R_2 = 6$$
$$X_1 + 2X_2 + S_2 = 4$$
$$X_1, X_2, S_1, S_2, R_1, R_2 \geq 0$$

Where $M > 0$ is some very large constant.

Initial solution $R_1 = 3,\ R_2 = 6,\ S_2 = 4.$

	X_1	X_2	S_1	R_1	R_2	S_2	b
Z	-4	-1	0	-M	-M	0	**0**
R_1	3	1	0	1	0	0	**3**
R_2	4	3	-1	0	1	0	**6**
S_2	1	2	0	0	0	1	**4**

Eliminate M's from R_1 and R_2 columns by pivot operations.

Initial tableau (now in canonical form):

	X_1	X_2	S_1	R_1	R_2	S_2	b
Z	**-4 + 7M**	**-1 + 4M**	**- M**	**0**	**0**	**0**	**9M**
R_1	**3**	**1**	**0**	**1**	**0**	**0**	**3**
R_2	**4**	**3**	**-1**	**0**	**1**	**0**	**6**
S_2	**1**	**2**	**0**	**0**	**0**	**1**	**4**

First iteration:

	X_1	X_2	S_1	R_1	R_2	S_2	b
Z	**0**	$\frac{(1 + 5M)}{3}$	**- M**	$\frac{(4 - 7M)}{3}$	**0**	**0**	**4 + 2M**
X_1	**1**	**1/3**	**0**	**1/3**	**0**	**0**	**1**
R_2	**0**	**5/3**	**-1**	**-4/3**	**1**	**0**	**2**
S_2	**0**	**5/3**	**0**	**-1/3**	**0**	**1**	**3**

Second iteration:

	X_1	X_2	S_1	R_1	R_2	S_2	b
Z	0	0	1/5	8/5 - M	-1/5 – M	0	18 / 5
X_1	1	0	1/5	3/5	-1/5	0	3/5
X_2	0	1	-3/5	-4/5	3/5	0	6/5
S_2	0	0	1	1	-1	1	1

Third iteration:

	X_1	X_2	S_1	R_1	R_2	S_2	b
Z	0	0	0	7/5 - M	– M	-1/5	17 / 5
X_1	1	0	0	2/5	0	-1/5	2/5
X_2	0	1	0	-1/5	0	3/5	9/5
S_1	0	0	1	1	-1	1	1

Optimum.

Note: If artificial variables remain in basis solution is infeasible.