Exercises on Matrices

1. Find the determinants and eigenvalues of the following matrices:
   
   (a) \[
   \begin{bmatrix}
   2 & 5 \\
   -1 & -1
   \end{bmatrix}
   \]
   
   (c) \[
   \begin{bmatrix}
   1 & 2 \\
   4 & 3
   \end{bmatrix}
   \]
   
   (b) \[
   \begin{bmatrix}
   7 & 3 \\
   3 & -1
   \end{bmatrix}
   \]
   
   (d) \[
   \begin{bmatrix}
   2 & 1 \\
   0 & -1
   \end{bmatrix}
   \]

2. With the eigenvalues, we can find the eigenvectors of a matrix. An Eigenvector of a matrix \( A \) is any solution vector \( x \) for which: \( Ax = \lambda x \).

Example: Find the eigenvalues and eigenvectors of the matrix \( \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \)

(a) Firstly find the eigenvalues:
   Recall that the eigenvalues are calculated by solving \( \det(A - \lambda I) = 0 \) (where \( I \) is the identity matrix). Thus
   \[
   A - \lambda I = \begin{bmatrix}
   1 - \lambda & -2 \\
   1 & 4 - \lambda
   \end{bmatrix}
   \]
   \[
   \det \begin{bmatrix}
   1 - \lambda & -2 \\
   1 & 4 - \lambda
   \end{bmatrix} = 0 \]
   gives the quadratic \((1 - \lambda)(4 - \lambda) + 2 = 0\) which simplifies to \(\lambda^2 - 5\lambda + 6 = 0\) hence \(\lambda = 2, 3\) are the eigenvalues.

(b) Now to find the eigenvectors: From above, these are the solution vectors to the system \( Ax = \lambda x \) when we substitute the eigenvalues above. Hence for \(\lambda = 2\), we get:
   \[
   \begin{bmatrix}
   1 & -2 \\
   1 & 4
   \end{bmatrix} \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix} = 2 \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix}
   \]
   So the first equation reads: \(x_1 - 2x_2 = 2x_1\) giving \(x_1 = -2x_2\) and thus the eigenvector \(x\) corresponding to the first eigenvalue \(\lambda\) is any multiple of \(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\).

   In a similar way, we can find the second eigenvector as any multiple of \(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\).

Find the eigenvectors of the above matrices in Q1 using the method outlined above.