Chapter 2:

Discrete Models

Glossary of Terms

Here are some of the types of symbols you will see in the Module:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol Face</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector</td>
<td>Lower-case bold</td>
<td>u, v, p</td>
</tr>
<tr>
<td>Matrix</td>
<td>Upper-case bold</td>
<td>M, X, A</td>
</tr>
<tr>
<td>Vector at Time step</td>
<td>Subscript</td>
<td>u₀</td>
</tr>
<tr>
<td>Age Category at Time step</td>
<td>Subscript &amp; Superscript</td>
<td>u₀</td>
</tr>
</tbody>
</table>
**Intro to the Topic**

**Discrete Models**

**Growth and Decay**

**Linear & Non-Linear Interaction Models**

**First Order Linear Difference Equations**

- We start with the most basic equations.
- State at time $t$ purely related to that at $t - 1$
- Example in nature is cell division
  \[ M_{n+1} = aM_n \]  
  \( a \) constant, \( n \) is the generation number
- So number in \( n \)th generation related to that in first generation by:
  \[ M_n = aM_{n-1} = \ldots = a^nM_0 \]  
- So if
  1. \( |a| > 1 \) the population will increase,
  2. \( |a| = 1 \) the population will be stable,
  3. \( |a| < 1 \) the population will decrease.

**Higher Order Linear Difference Equations**

Example 1: Rabbit Reproduction

- **Order** of difference equation is number of terms determining present state.
- Examples of higher order difference eqns common in nature.
- Leonardo of Pisa (*Fibonacci*) modelled rabbit reproduction.
- Assumptions of Fibonacci model:
  - Each pair of rabbits can reproduce from two months old
  - Each reproduction produces only one pair of rabbits
  - All rabbits survive.
- Number of rabbit pairs at time \( n + 1 \), \( M_{n+1} \) (for \( n \) months) given by:
  \[ M_{n+1} = M_n + M_{n-1}. \]  
- With \( M_0 = 1 \), \( M_1 = 1 \), (1 pair to start) number grows as 1, 1, 2, 3, 5, 8, 13, . . .
Example 1: Rabbit Reproduction (cont’d)

Rather than Eqn.(2.3), 'one step' eqn (like Eqn.(2.1)) is better.

Get this by writing Eqn.(2.3) in the form:

\[ \begin{align*}
  M_{n+1} + M_n &= M_{n+2} \\
  M_{n+1} &= M_{n+1} + M_{n+2}
\end{align*} \]  \hspace{1cm} (2.4)

which, by writing

\[ u_n = \begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix} \]

takes the form

\[ u_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} u_n. \]  \hspace{1cm} (2.5)
Digression: Matrix Basics

Matrices & Vectors

- A matrix is an array of coefficients of the form:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \quad (2.6)
\]

- This is called an $m \times n$ matrix as it has $m$ rows and $n$ columns.

- A vector is an array of coefficients of the form:

\[
A = \begin{pmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{pmatrix}
\]

Digression: Matrix Basics cont’d

Matrix Systems

- In the course we will see systems of equations of the form:

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*} \quad (2.7)
\]

for a system of two equations in two unknowns $x_1, x_2$ with constant coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ and a right-hand side $b_1, b_2$.

- With matrix multiplication, this can be written as:

\[
Ax = b \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (2.8)
\]
Matrix Inverse, Identity Matrix

- It can be shown (c.f. Strang), that Eqn.(2.8) has a unique solution if the inverse of the matrix exists.
- The inverse of the matrix $A^{-1}$ has the property:
  \[ A \times A^{-1} \text{ is the Identity Matrix } I \]

- The $n \times n$ identity matrix is given by:
  \[ I = \begin{pmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1
\end{pmatrix} \tag{2.9} \]

Solutions to Matrix Systems: Matrix Determinant

- To solve $x = (x, y)$ in Eqn.(2.8) need to find $A^{-1}$
- For a $2 \times 2$ matrix $A^{-1}$ is given by:
  \[ A^{-1} = \frac{1}{\det(A)} \begin{pmatrix}
  a_{22} & -a_{12} \\
  -a_{21} & a_{11}
\end{pmatrix} \tag{2.10} \]
  where $\det(A)$ is the determinant of the matrix $A$
- The determinant of $A$ is given by $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
- Eqn.(2.10) holds for a $2 \times 2$ matrix only.
- The solution to the linear system in Eqn.(2.8) will only exist if the following condition is met:
  \[ \det(A) \equiv \begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix} \neq 0 \tag{2.11} \]
Matrix Characteristic Equation, Trace

- The characteristic equation for $A$ is given by $\det(A - \lambda I) = 0$.
- It arises in a number of circumstances, as we shall see later.
- For a $2 \times 2$ matrix, this expression becomes:

$$\det(A - \lambda I) = 0 \equiv \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \quad (2.12)$$

which reduces to $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$
which we rewrite as

$$\lambda^2 - p\lambda + q = 0 \quad (2.13)$$

where $p = a_{11} + a_{22}$ is called the Trace of $A$ and $q = \det(A)$.

Matrix Eigenvalues, Eigenvectors

- The roots of the quadratic equation in Eqn.(2.13) are given by:

$$\lambda_{1,2} = \frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2} \quad (2.14)$$

are known as the eigenvalues of $A$.
- It can be shown (see again Strang) that any matrix $A$ can be decomposed as follows:

$$A = S\Lambda S^{-1} \quad (2.15)$$

where $S$ has eigenvectors of $A$, $v_1, v_2$ on the columns, & $\Lambda$ is a matrix with eigenvalues as diagonals & zeros elsewhere:

$$A = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S^{-1} \quad (2.16)$$
Eqn. (2.15) comes in useful (amongst other things) for raising matrices to powers:

\[ A^3 = (SΛS^{-1})(SΛS^{-1})(SΛS^{-1}) = (SΛ^3S^{-1}) \quad (2.17) \]

The eigenvectors \( v_1, v_2 \) are the solutions to the linear system \( Ax = λx \) for \( λ = λ_1, λ_2 \) respectively.

As with eigenvalues, these have important physical meanings for the system under consideration.

The process in Eqn. (2.15) is known as \textit{eigen decomposition} for a square matrix; where the matrix is not square, it is known as \textit{singular value decomposition}.

For difference equations the system at time step \( n \) is related to that at the previous step \( n - 1 \) through the system:

\[ u_n = Au_{n-1} = A^n u_0 \quad (2.18) \]

Using eigendecomposition \( A = SΛS^{-1} \) and setting

\[ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S^{-1}u_0 = S^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} _{n=0} \]

we observe that

\[ u_n = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c_1 λ_1^n v_1 + c_2 λ_2^n v_2 \quad (2.19) \]

where \( c_1, c_2 \) are constants.
Matrix Decomposition, Difference & Differential Equations

A similar result may be obtained for differential equations where the system of a second order equation (often) has a solution of the form:

\[
x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.
\]  

(2.20)

So the solutions of difference and differential equations can be broken down into a **linear combination** of the eigenvalues and corresponding eigenvectors of the original matrix system.

This is a very important result and, as we will see, comes in very useful for determining dominant or longterm solutions of matrix systems such as the Fibonacci series.

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Eigenvalues and the Fibonacci Difference Equation

In order to find the long-term behaviour of the Fibonacci system in Eqn.(2.5), we can write (using Eqn.(2.17))

\[
u_n = A^n u_0 = S \Lambda^n S^{-1} u_0\]

(2.21)

Given that

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

from Eqn.(2.5), we find the characteristic equation to be

\[\lambda^2 - \lambda - 1 = 0\]

(from Eqn.(2.12)).

This gives the eigenvalues

\[
\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.
\]
Stability of Fibonacci Sequences

- The full eigendecomposition for $A$ can then be found to be

\[
A = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix}
\lambda_1 & \lambda_2 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
1 & -\lambda_2 \\
-1 & \lambda_1
\end{pmatrix}
\]  

(2.22)

Thus Eqn.(2.21) reduces to

\[
\begin{pmatrix}
M_{n+1} \\
M_n
\end{pmatrix} = S \begin{pmatrix}
\lambda_1^n & 0 \\
0 & \lambda_2^n
\end{pmatrix} S^{-1} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]  

(2.23)

- The $n$th Fibonacci number is 2nd element of vector on left hand side of Eqn.(2.23). $M_n$, can be shown to be:

\[
M_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]  

(2.24)

Stability of Fibonacci Sequences cont’d

The Golden Number & Fibonacci Sequences

- $\phi = (1 + \sqrt{5})/2$ is very important and was known to the Ancient Greeks as the golden number because rectangles with sides in the ratio $1 : 1.618$ were the most elegant.
- The Golden Number occurs frequently in nature and persists in the design of everyday items such as credit cards, ipods etc.
- As $\lambda_2 > 1$ & $-1 < \lambda_1 < 0$, $\lambda_2$ is the largest eigenvalue and its magnitude means the Fibonacci sequence is monotonically increasing.
- The fact that $\lambda_1$ is negative and of magnitude less than 1 means it contributes a slight oscillation that dies out as $n$ increases. This can be seen in Fig. 2.2.
Example 2: Pig Reproduction

- A pair of bonhams becomes a mature pair of pigs in the next season.
- A mature pair produces six pairs of bonhams in the following season, and in every successive season thereafter.
- Each pair of bonhams produced takes one season to reach maturity and a further season to start breeding (and producing six young pairs) in every subsequent season.
- This can be seen in the diagram (fig 2.3).
- It is assumed that breeding is seasonal so that generations do not overlap and that pigs live a long time.
As with eqn(2.5), we may derive an expression for number of pairs of pigs in the \( n + 1 \)th generation w.r.t. the \( n \)th generation:

\[
\mathbf{u}_{n+1} = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix} \mathbf{u}_n.
\] (2.25)

which (from eqn(2.12)) leads to the eigenvalue problem:

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \equiv \begin{vmatrix} 1 - \lambda & 6 \\ 1 & 0 - \lambda \end{vmatrix} = 0
\] (2.26)

which reduces to

\[
\lambda^2 - \lambda - 6 = 0
\] (2.27)

giving eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = -2 \).
The full eigendecomposition can then be found to be:

\[
A = \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}
\]  

(2.28)

Thus, as in eqn(2.18) above for the Fibonacci example:

\[
u_n = A u_{n-1} = A^n u_0
\]  

(2.29)

Which may be shown to be:

\[
u_n = \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \left( \begin{pmatrix} 3^n & 0 \\ 0 & -2^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \right) u_0
\]  

(2.30)

Which reduces to

\[
u_n = \frac{1}{5} \begin{bmatrix} 3(3^n) + 2(-2)^n \\ 3^n - (-2)^n \end{bmatrix}
\]  

(2.31)

for an initial population \( u_0 = (1, 0)^T \) (i.e. one breeding pair).

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A Small Wager

Example 3: A Small Wager

A cautious but enthusiastic sporting fan decides to speculate on their team winning consecutive weekly matches.

Starting at Week 1 with €1, they put a bet at 1.05 (i.e. \( \frac{21}{20} \)) on the previous week’s winnings plus a bet at 1.1 (i.e. \( \frac{11}{10} \)) on the week before’s.

Assuming that the fan is successful every week, calculate how their winnings accumulate.

So, following Eqn.(2.3), if amount at week \( n + 1 \) is given by \( M_{n+1} \):

\[
M_{n+1} = 1.05M_n + 1.1M_{n-1}.
\]  

(2.32)

With \( M_0 = 0, M_1 = 1 \)
A Small Wager (cont’d)

Hence

\[ 1.05M_{n+1} + 1.1M_n = M_{n+2} \]

which, by writing (as with the Rabbit Reproduction Example above)

\[ u_n = \begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix} \]

takes the form

\[ u_{n+1} = \begin{pmatrix} 1.05 & 1.1 \\ 1 & 0 \end{pmatrix} u_n, \quad \text{with} \quad u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

Thus \( u_1 = \begin{pmatrix} 1.05 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 2.2025 \\ 1.05 \end{pmatrix} \) etc.

As previously (with eqn(2.5) and eqn(2.25)), we may derive an expression for the amount in the \( n+1 \)th week w.r.t. the \( n \)th week:

\[ u_{n+1} = \begin{pmatrix} 1.05 & 1.1 \\ 1 & 0 \end{pmatrix} u_n. \]  \hspace{1cm} (2.35)

which (from eqn(2.12)) leads to the eigenvalue problem:

\[ \det(A - \lambda I) = 0 \equiv \begin{vmatrix} 1.05 - \lambda & 1.1 \\ 1 & 0 - \lambda \end{vmatrix} = 0 \]  \hspace{1cm} (2.36)

which reduces to

\[ \lambda^2 - 1.05\lambda - 1.1 = 0 \]  \hspace{1cm} (2.37)

giving eigenvalues \( \lambda_1 = 1.7 \) and \( \lambda_2 = -0.65 \).
Thus, using Eqn(2.17) above

\[ A^3 = (S\Lambda S^{-1}) (S\Lambda S^{-1}) (S\Lambda S^{-1}) = (S\Lambda^3 S^{-1}), \quad (2.38) \]

the full eigendecomposition can then be found to be:

\[ A = \begin{pmatrix} 1.7 & -0.65 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.7 & 0 \\ 0 & -0.65 \end{pmatrix} \begin{pmatrix} 0.43 & 0.28 \\ -0.43 & 0.78 \end{pmatrix} \]

(2.39)

Thus, as in eqn(2.18) above for the Fibonacci example:

\[ u_n = A_{n-1}u = A^n u_0 \]

(2.40)

Which may be shown to be:

\[ u_n = \begin{pmatrix} 1.7 & -0.65 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.7^n & 0 \\ 0 & (-0.65)^n \end{pmatrix} \times \begin{pmatrix} 0.43 & 0.28 \\ -0.43 & 0.78 \end{pmatrix} u_0 \]

Which reduces to

\[ u_n = 0.43 \left[ \begin{pmatrix} 1.7^{n+1} \end{pmatrix} - \begin{pmatrix} (-0.65)^{n+1} \end{pmatrix} \right] \]

(2.41)

for an initial sum of \( u_0 = (1, 0)^T \), this gives \( u_1 \approx \begin{pmatrix} 1.05 \\ 1 \end{pmatrix} \) etc.