Pushdown Automata

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CA215 - Languages and Computability

November 23rd, 2009

Extending Finite State Automata

Not every context-free language can be recognized by a FSA.

Can we extend the FSA concept to create FSA-like machines that recognize a language iff it is a context-free language?

Consider $L = \{wcw^r | w \in \Sigma^*\}$
An automaton that recognizes this must "remember" w until it sees c and then checks for w^r (w reversed).

This can be done with a stack memory.

Pushdown Automata

Consider a NFA with stack memory. This is called a *pushdown automaton* (PDA).

Recall that, for any RG rule $A \to aB$, a finite automaton can imitate it by reading a in state A and going to state B.

Pushdown Automata

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Recall that, for any RG rule $A \rightarrow aB$, a finite automaton can imitate it by reading a in state A and going to state B.

Similarly, for any CFG rule $A \rightarrow aBb$, a pushdown automaton can imitate it by reading a in state A, putting b on the stack, and going to state B.

Pushdown Automata

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Recall that, for any RG rule $A \rightarrow aB$, a finite automaton can imitate it by reading a in state A and going to state B.

Similarly, for any CFG rule $A \rightarrow aBb$, a pushdown automaton can imitate it by reading a in state A, putting b on the stack, and going to state B.

The PDA can decide (non-deterministically) to pop values off the stack and compare against the next input characters.

Theorem: A language is context-free iff it can be recognized by a pushdown automaton.

PDA Definition

A pushdown automaton is a 6-tuple $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$, where:

- Q is a finite set of *states*,
- Σ is an alphabet (*input symbols*),
- Γ is an alphabet (*stack symbols*),
- $q_0 \in Q$ is the *initial state*,
- $F \subseteq Q$ is the set of *final states*,
- δ is the *transition function*, $Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \to Q \times (\Gamma \cup \{\epsilon\})$.

If $((q, a, \alpha), (q', \beta)) \in \delta$, then when in state q with α at the top of the stack, M may

- read a from input
- replace α with β on the top of the stack
- enter state q'

PDA Definition

Notation:
(q, a, α) : a *configuration* (the state of the machine, the portion of the input yet to be read, the contents of the stack, read top-down)

$(q, a, \alpha) \vdash (q', b, \beta)$: (q, a, α) *yields* (q', b, β) in one step

$(q, a, \alpha) \vdash^* (q', b, \beta)$: (q, a, α) yields (q', b, β) in *zero or more* steps

M accepts $w \in \Sigma^*$ iff $(q_0, w, \epsilon) \vdash_M^* (q', \epsilon, \epsilon)$ for $q' \in F$.

Push: $((q, a, \epsilon), (q', \beta))$ pushes β onto the stack.
Pop: $((q, a, \alpha), (q', \epsilon))$ pops α from the stack.

PDA Example 1

$L = \{wcw^r : w \in \{a, b\}^*\}$ (palindromes of odd length)

Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$
\begin{aligned}
Q &= \{s, f\} \\
\Sigma &= \{a, b, c\} \\
\Gamma &= \{a, b\} \\
q_0 &= s \\
F &= \{f\} \\
\delta &= \begin{array}{l}
((s, a, \epsilon), (s, a)) \\
((s, b, \epsilon), (s, b)) \\
((s, c, \epsilon), (f, \epsilon)) \\
((f, a, a), (f, \epsilon)) \\
((f, b, b), (f, \epsilon))
\end{array}
\end{aligned}
$$

PDA Example 1

$L = \{wcw^r : w \in \{a, b\}^*\}$ (palindromes of odd length)
Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$\delta = \begin{array}{ll} ((s, a, \epsilon), (s, a)) & (1) \\ ((s, b, \epsilon), (s, b)) & (2) \\ ((s, c, \epsilon), (f, \epsilon)) & (3) \\ ((f, a, a), (f, \epsilon)) & (4) \\ ((f, b, b), (f, \epsilon)) & (5) \end{array}$$

Example: *abbcbba*
$(s, abbcbba, \epsilon) \vdash (s, bbcbba, a) \vdash (s, bcbba, ba) \vdash (s, cbba, bba) \vdash$
$(f, bba, bba) \vdash (f, ba, ba) \vdash (f, a, a) \vdash (f, \epsilon, \epsilon)$

acb
$(s, acb, \epsilon) \vdash (s, cb, a) \vdash (f, b, a) \vdash ??$

PDA Example 2

$L = \{ww^r : w \in \{a, b\}^*\}$ (palindromes of even length)
Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$
\begin{aligned}
Q &= \{s, f\} \\
\Sigma &= \{a, b\} \\
\Gamma &= \{a, b\} \\
q_0 &= s \\
F &= \{f\} \\
\delta &= \begin{array}{l}
((s, a, \epsilon), (s, a)) \\
((s, b, \epsilon), (s, b)) \\
((s, \epsilon, \epsilon), (f, \epsilon)) \\
((f, a, a), (f, \epsilon)) \\
((f, b, b), (f, \epsilon))
\end{array}
\end{aligned}
$$

PDA Example 2

$L = \{ww^r : w \in \{a,b\}^*\}$ (palindromes of even length)
Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$\delta = \begin{array}{ll} ((s,a,\epsilon),(s,a)) & (1) \\ ((s,b,\epsilon),(s,b)) & (2) \\ ((s,\epsilon,\epsilon),(f,\epsilon)) & (3) \\ ((f,a,a),(f,\epsilon)) & (4) \\ ((f,b,b),(f,\epsilon)) & (5) \end{array}$$

Just like previous example, except for Transition 3.
Example: *abba*
$(s, abba, \epsilon) \vdash (s, bba, a) \vdash (s, ba, ba) \vdash (f, ba, ba) \vdash (f, a, a) \vdash (f, \epsilon, \epsilon)$

PDA Example 3

$L = \{w \in \{a, b\}^* : w$ has an equal number of a's and b's$\}$
Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$
\begin{aligned}
Q &= \{s, q, f\} \\
\Sigma &= \{a, b\} \\
\Gamma &= \{a, b, c\} \\
q_0 &= s \\
F &= \{f\} \\
\delta &= \begin{matrix} ((s, \epsilon, \epsilon), (q, c)) \\ ((q, a, \epsilon), (q, a)) \\ ((q, a, b), (q, \epsilon)) \\ ((q, b, \epsilon), (q, b)) \\ ((q, b, a), (q, \epsilon)) \\ ((q, \epsilon, c), (f, \epsilon)) \end{matrix}
\end{aligned}
$$

c is a special symbol to mark the bottom of the stack.
Keep either the excess a's or excess b's in the stack.

PDA Example 3

$L = \{w \in \{a,b\}^* : w \text{ has an equal number of } a\text{'s and } b\text{'s}\}$
Let $M = (Q, \Sigma, \Gamma, q_0, F, \delta)$

$$\delta = \begin{array}{ll} ((s,\epsilon,\epsilon),(q,c)) & (1) \\ ((q,a,\epsilon),(q,a)) & (2) \\ ((q,a,b),(q,\epsilon)) & (3) \\ ((q,b,\epsilon),(q,b)) & (4) \\ ((q,b,a),(q,\epsilon)) & (5) \\ ((q,\epsilon,c),(f,\epsilon)) & (6) \end{array}$$

Example: *aabbba*

$(s, aabbba, \epsilon) \vdash (q, aabbba, c) \vdash (q, abbba, ac) \vdash (q, bbba, aac) \vdash (q, bba, ac) \vdash (q, ba, c) \vdash (q, a, bc) \vdash (q, \epsilon, c) \vdash (f, \epsilon, \epsilon)$

aa

$(s, aa, \epsilon) \vdash (q, aa, c) \vdash (q, a, ac) \vdash (q, \epsilon, aac) \vdash ??$