AXIOMS OF PROBABILITY

A probability function P is defined on subsets of the sample space **S** to satisfy the following axioms:

1. Non-Negative Probability:

$$P(E) \geq 0.$$

2. Mutually-Exclusive Events:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

provided E_1 and E_2 are mutually exclusive. i.e. $E_1 \cap E_2$ is empty.

3. The Universal Set:

$$P(S) = 1$$

Properties of Probability

Theorem 1: Complementary Events

For each $E \subset S$:

$$P(\overline{E}) = 1 - P(E)$$

Proof:

$$S = E \cup \overline{E}$$

Now, E and $\overline{E}$ are mutually exclusive.
i.e.

$$E \cap \overline{E} \text{ is empty.}$$

Hence:

$$P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E})$$

(Axiom 2)

Also:

$$P(S) = 1$$

(Axiom 3)

i.e.

$$\begin{aligned} P(S) &= P(E) + P(\overline{E}) \\ \longrightarrow \quad 1 &= P(E) + P(\overline{E}) \end{aligned}$$

So:

$$P(\overline{E}) = 1 - P(E)$$

Properties of Probability

Theorem 2: The Impossible Event/The Empty Set

$$P(\emptyset) = 0 \text{ where } \emptyset \text{ is the empty set}$$

Proof:

$$S = S \cup \emptyset$$

Now: S and $\emptyset$ are mutually exclusive.
i.e.

$$S \cap \emptyset \text{ is empty.}$$

Hence:

$$P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$$

(Axiom 2)

Also:

$$P(S) = 1$$

(Axiom 3)

i.e.

$$1 = 1 + P(\emptyset)$$

i.e.

$$P(\emptyset) = 0.$$

Properties of Probability

Theorem 3:

If E_1 and E_2 are subsets of S such that $E_1 \subset E_2$, then

$$P(E_1) \leq P(E_2)$$

Proof:

$$E_2 = E_1 \cup (\overline{E}_1 \cap E_2)$$

Now, since E_1 and $\overline{E}_1 \cap E_2$ are mutually exclusive,

$$\begin{aligned} P(E_2) &= P(E_1) + P(\overline{E}_1 \cap E_2) Axiom 2 \\ &\geq P(E_1) \end{aligned}$$

since $P(\overline{E}_1 \cap E_2) \geq 0$ from Axiom 1.

Properties of Probability

Theorem 4: Range of Probability

For each $E \subset S$

$$0 \leq P(E) \leq 1$$

Proof:
Since,

$$\emptyset \subset E \subset S$$

then from Theorem 3,

$$P(\emptyset) \leq P(E) \leq P(S)$$

$$0 \leq P(E) \leq 1$$

Theorem 5: The Addition Law of Probability

If E_1 and E_2 are subsets of S then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof:

$$E_1 \cup E_2 = E_1 \cup (E_2 \cap \overline{E}_1)$$

Now, since E_1 and $E_2 \cap \overline{E}_1$ are mutually exclusive,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2 \cap \overline{E}_1) \qquad (1)$$

(Axiom 2)

Now E_2 may be written as two mutually exclusive events as follows:

$$E_2 = (E_2 \cap E_1) \cup (E_2 \cap \overline{E}_1)$$

So

$$P(E_2) = P(E_2 \cap E_1) + P(E_2 \cap \overline{E}_1)$$

(Axiom 2)

Thus:

$$P(E_2 \cap \overline{E}_1) = P(E_2) - P(E_2 \cap E_1) \qquad (2)$$

Inserting (2) in (1), we get

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Example:

Of 200 employees of a company, a total of 120 smoke cigarettes: 60% of the smokers are male and 80% of the non smokers are male. What is the probability that an employee chosen at random:

1. is male or smokes cigarettes

2. is female or does not smoke cigarettes

3. either smokes or does not smoke