Bayes Theorem

Randomised Response

Bayes Theorem

An important branch of applied statistics called Bayes Analysis can be developed out of conditional probability.

It is possible given the outcome of the second event in a sequence of two events to determine the probability of various possibilities for the first event.

Example – How to buy a used car

I am thinking of buying a used car at Honest Ed's. In order to make an informed decision, I look up the records in an auto magazine of the car type I am interested in and find that unfortunately 30% have faulty transmissions.

To get more information on this particular car at Honest Ed's I hire a mechanic who can make a shrewd guess on the basis of a quick drive around the block. Of course, he isn't always right but he does have an excellent record. Of all the faulty cars he has examined in the past, he correctly pronounced 90% "faulty". In other words, he wrongly pronounced only 10% "ok".

He has almost as good a record in judging good cars. He has correctly pronounced 80%"ok", while he wrongly pronounced only 20% "faulty".

- "faulty" describes the mechanics opinion.
- faulty with no quotation marks describes the actual state of the car.

Example – How to buy a used car

What is the chance that the car I'm thinking of buying has a faulty transmission:

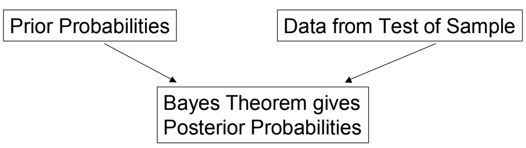
1. Before I hire the mechanic?
2. If the mechanic pronounces it "faulty"?
3. If the mechanic pronounces it "ok"?

The Logic of Bayes Theorem

Bayes Theorem gives us a way of calculating P(A|B) from a knowledge of P(B|A).

The point of Bayes may be stated more generally:

Prior probabilities combined with some sort of information such as a test or sample yield posterior probabilities.



Posterior Probability

Law of Total Probability

If a sample space can be partitioned into *k* mutually exclusive and exhaustive events

$A_1, A_2, A_3, \ldots A_k$

i.e. $S = A_1 \cup A_2 \cup A_3 \ldots \cup A_k$

Then for any event E:

$P(E) = P(A_1)P(E|A_1) + P(A_2)P(E|A_2) \ldots + P(A_k)P(E|A_k)$

Proof:

$E = E \cap S$

$= E \cap (A_1 \cup A_2 \cup \ldots A_k)$

$= (E \cap A_1) \cup (E \cap A_2) \cup \ldots (E \cap A_k)$

Since these are mutually exclusive

$P(E) = P(E \cap A_1) + P(E \cap A_2) + \ldots P(E \cap A_k)$

$= P(A_1)P(E|A_1) + P(A_2)P(E|A_2) \ldots + P(A_k)P(E|A_k)$

Bayes Theorem

If a sample space can be partitioned into *k* mutually exclusive and exhaustive events $A_1, A_2, A_3, \ldots A_k$

$$S = A_1 \cup A_2 \cup A_3 \ldots \cup A_k$$

Then for any event E: $P(A_i \mid E) = \frac{P(A_i)\, P(E|A_i)}{P(E)}$

where

$P(E) = P(A_1)P(E|A_1) + P(A_2)P(E|A_2) \ldots + P(A_k)P(E|A_k)$

Proof:

For any *i*, $1 \le i \le k$

$= E \cap A_i \quad = \quad A_i \cap E$

$= P(E)\, P(A_i|E) = P(A_i)\, P(E|A_i)$

Prior Probability

Posterior Probability → $P(A_i \mid E) = \frac{P(A_i)\, P(E|A_i)}{P(E)}$ ← **Conditional Probability**

Relating Bayes To The Car Example

Prior probabilities

The initial probabilities before any testing are called the prior probabilities.

Posterior probabilities

The probabilities after testing are called the posterior probabilities.

Note that the sum of the branches from each node sum to 1.

Example

An insurance company runs three different offices , A, B and C. 30% of the company's employees work in Office A, 20% in Office B and 50% in Office C.
10% of the staff in Office A are managers, 20% of the staff in Office B are managers and 5% from Office C are managers.

Offices	A	B	C
Proportion Employees	.3	.2	.5
Proportion Managers	.1	.2	.05

(a) What is the total proportion of managers in the company?

(b) If a member of staff, chosen at random, turns out to be a manager, what is the probability that she works in Office A?

Randomised Response

It was anticipated that it would not be possible to assess the extent of tax evasion by direct questioning, so it was decided to use the randomised response technique. Each respondent was given a card with the following two questions:

1. Was your mother born in April?
2. Have you ever evaded tax?

Then the respondent was asked to toss a coin and answer (1) if a head turned up and (2) for a tail. The interviewer did not know which question was answered. Out of 1,000 people interviewed, 200 answered yes. Estimate the proportion of tax evaders.