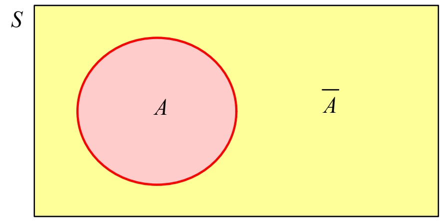
Conditional Probability

- Review of Events.
- Addition rule.
- Conditional probability.
- Multiplication law
- Independent events.



S = Sample Space.
P(S) = 1

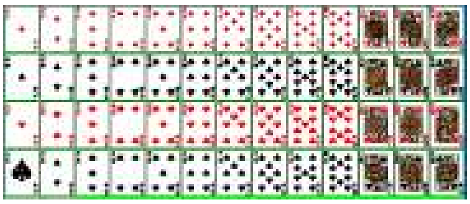
ø = null, impossible event = { } null event cannot occur
P(ø) = 0

Complement of Event A, Ā = all outcomes ***not*** in A.

Example: Draw a card from a well shuffled pack.

A = event of drawing an Ace

Ā = event of not drawing an Ace



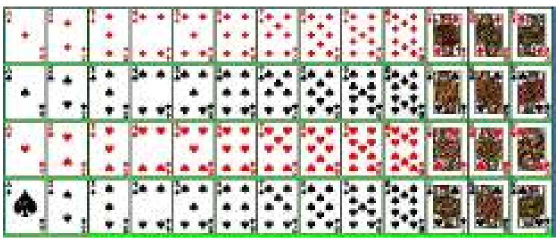
Probability of Event A Occurring

Assumes all outcomes of the experiment are equally likely.

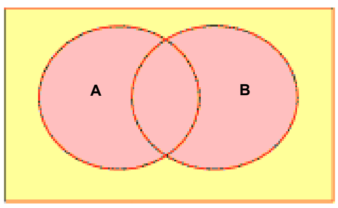
$$P(A) = \frac{\text{Number of Favourable Outcomes in } A}{\text{Total Number of Outcomes in } S}$$

Example: Draw a card from a well shuffled pack.
A = event of drawing an Ace
Ā = event of NOT drawing an Ace

P(A) = $\frac{4}{52}$ P(Ā) = 1 – P(A) = 1 - $\frac{4}{52} = \frac{48}{52}$



Union of Events A and B
A or B
AUB



Shaded area gives the union of events *A* and *B*

A ∪ Ā = S

MUTUALLY EXCLUSIVE EVENTS

Definition

Events that cannot occur together are said to be ***mutually exclusive events***.

Example

Consider the following events for one roll of a die:

A= an even number is observed= {2, 4, 6}

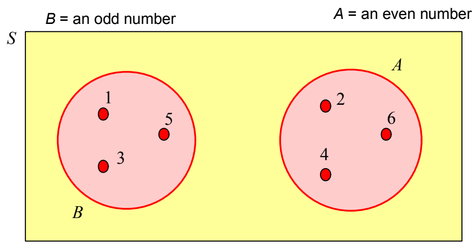
B= an odd number is observed= {1, 3, 5}

C= a number less than 5 is observed= {1, 2, 3, 4}

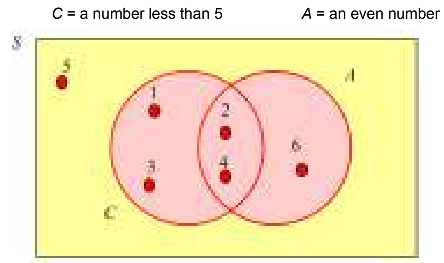
Are events *A* and B mutually exclusive?

Are events *A* and *C* mutually exclusive?

Mutually exclusive events *A* and *B*.



Mutually nonexclusive events *A* and *C*.



General Addition Rule

- General Addition Rule:
  For any two events **A** and **B**,
  P(**A** *or* **B**)
  = *P*(**A** U B) = *P*(**A**) + *P*(**B**) – *P*(**A** *and* **B**).

- The following shows a situation in which we would use the general addition rule:

General Addition Rule to Find the Union of 2 Events

Draw a card: event A – an Ace; event B – a heart

	B	$\bar{B}$	Total
A	$\frac{1}{52}$		$\frac{4}{52}$
$\bar{A}$			
Total	$\frac{13}{52}$		1.00

What is the probability of getting either an Ace or a Heart

P(**A** *or* **B**) = *P*(**A**) + *P*(**B**) – *P*(**A** *and* **B**)

P(**Ace** *or* **Heart**)
= *P*(**Ace**) + *P*(**Heart**) – *P*(**Ace** *and* **Heart**)
= 4/52 + 13/52 – 1/52
= 16/52
= 4/13

Addition Rule for Two Mutually Exclusive (or disjoint) Events A and B:

The probability that one *or* the other occurs is the sum of the probabilities of the two events.

P(**A** U **B**) = *P*(**A** *or* **B**) = *P*(**A**) + *P*(**B**)

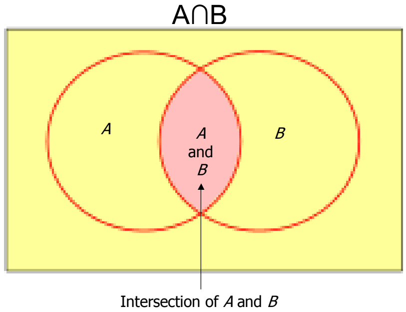
For mutually exclusive events A, B , C
P(A U B U C) = P(A) + P(B) + P(C)
etc, for more events

What is the probability of getting either a Spade or a Heart

P(**A** *or* **B**) = *P*(**A**) + *P*(**B**)

P(**Spade** *or* **Heart**)
= *P*(**Spade**) + *P*(**Heart**)
= 13/52 + 13/52
= 26/52
= 13/26

Intersection of Events A and B
A∩B

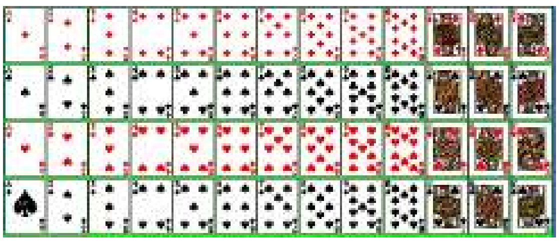


Example: Ace and a Spade i.e. Ace of Spade

Note for mutually exclusive events A∩B = ø

Conditional Probability

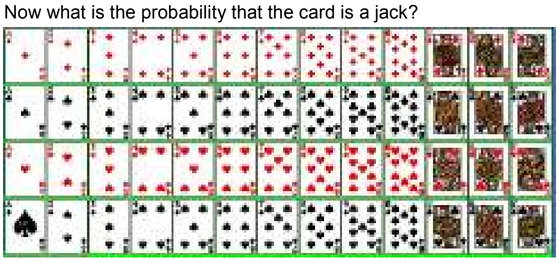
Suppose a card is drawn from a deck of cards.
What is the probability of drawing a jack?



4/52

Conditional Probability

Told card drawn is a face card.
Now what is the probability that the card is a jack?



Probability of a jack is now 4 / 12

Conditional Probability

Additional information or conditions generally change the probability of an event.

Let A = event chosen card is a jack.
B = event chosen card is a face card.
Looking for P(A|B)

Read as "given"

P (jack | face card)

The event whose probability is to be determined

This event has already occurred

P(jack | face card) = 4 / 12

MARGINAL AND CONDITIONAL PROBABILITIES cont.

Definition

Conditional probability is the probability that an event will occur given that another has already occurred. If *A* and *B* are two events, then the conditional probability *A* given *B* is written as

P (*A* | *B*)

and read as "the probability of *A* given that *B* has already occurred."

Conditional Probability

- To find the probability of the event **A** *given* the event **B**, we restrict our attention to the outcomes in **B**. We then find the fraction of *those* outcomes in **A** that also occurred.
- A probability that takes into account a given condition is called a **conditional probability**.

MARGINAL AND CONDITIONAL PROBABILITIES

Example

Suppose all 100 employees of a company were asked whether they are in favour of or against paying high salaries to CEOs of Irish companies. The following table gives a two way classification of the responses of these 100 employees.

MARGINAL AND CONDITIONAL PROBABILITIES

Two-Way Classification of Employee Responses with Totals

	In Favour	Against	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

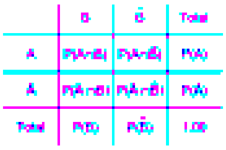
MARGINAL AND CONDITIONAL PROBABILITIES

Definition

Marginal probability is the probability of a single event without consideration of any other event. Marginal probability is also called ***simple probability***.

Listing the Marginal Probabilities

	In Favour (A)	Against (B)	Total	
Male (M)	15	45	60	$P(M) = 60/100 = .60$
Female (F)	4	36	40	$P(F) = 40/100 = .40$
Total	19	81	100	
	$P(A) = 19/100 = .19$	$P(B) = 81/100 = .81$		



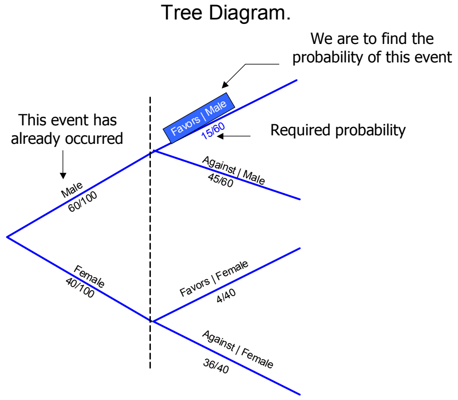
$P(A \cap M) = 15 / 100$

Example: Compute the conditional probability P (in favour | male) for the data on 100 employees given in the table.

	In Favour	Against	Total
Male	15	45	60
	↑ Males who are in favour		↑ Total number of males

$$P(\text{in favour} \mid \text{male}) = \frac{\text{Number of males who are in favour}}{\text{Total number of males}} = \frac{15}{60} = .25$$

Tree Diagram.



Calculating Conditional Probability

If A and B are two events, then,

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{and} \quad P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad \text{and} \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

given that $P(A) \neq 0$ and $P(B) \neq 0$.

A rearrangement of the above definition yields the following

Multiplication Law of Probability

Two events $P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)$

For more than 2 events:

$P(E_1 \cap E_2 \cap E_3 \cap E_{4 \ldots\ldots} \cap E_k) =$

$P(E_1)\, P(E_2 \mid E_1)\, P(E_3 \mid E_1 \cap E_2) \ldots P(E_k \mid E_1 \cap E_{2 \ldots} \cap E_{k-1})$

Terminology

Joint Probability: $P(A \cap B)$

Marginal Probability: $P(A)$, $P(B)$

Conditional Probability: $P(A|B)$ or $P(B|A)$

Example of Intersection

Probability that person is male and "in favour"

$P(M \cap A) = P(M) P(A|M) = P(A) P(M|A)$

	In Favour (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

P(M) = 60/100

P(A|M) = 15/60

$P(M \cap A) = P(M) P(A|M) = (60/100)(15/60) = 15/100 = .15$

P(A) = 19/100

P(M|A) = 15/19

$P(M \cap A) = P(A) P(M|A) = (19/100)(15/19) = 15/100 = .15$

Conditional Probability Calculation

	In Favour (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

P (in favour | male) = *P* (A | M)

$= P(A \cap M) / P(M) = (15/100) / (60/100) = 15/60 = .25$

Example of Multiplication Law for more than 2 Events

Pull three cards from a deck without replacement. What is the probability that all are black?

$P(E_1 \cap E_2 \cap E_3) = P(E_1) P(E_2 | E_1) P(E_3 | E_1 \cap E_2)$

$P(\text{all 3 cards are black}) = P(B1) P(B2 | B1) P(B3 | B1 \cap B2)$

$P(1^{st}$ card is black) = P(B1) = 26/52

$P(2^{nd}$ card is black given the 1^{st} card is black)= P(B2|B1)= 25/51

$P(3^{rd}$ card is black given that 1^{st} and 2^{nd} cards are black)

$= P(B3 | B1 \cap B2) = 24/50$

$P(\text{all 3 cards are black}) = P(B1) P(B2 | B1) P(B3 | B1 \cap B2)$

= (26/52) (25/51) (24/50) = 0.117

INDEPENDENT VERSUS DEPENDENT EVENTS

Definition

Two events are said to be ***independent*** if the occurrence of one does not affect the probability of the occurrence of the other. In other words, *A* and *B* are ***independent events*** if

either $P(A | B) = P(A)$ or $P(B | A) = P(B)$

Example of Independent Event

Throwing a dice twice.

The outcome of the first will not effect the outcome of the second throw.

Example

Are events "female (*F*)" and "in favour (*A*)" independent?

	In Favour (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

Events *F* and *A* will be independent if $P(F) = P(F | A)$

Otherwise they will be dependent.

From the information given in the Table

$P(F) = 40/100 = .40$

$P(F | A) = 4/19 = .2105$

Because these two probabilities are not equal, the two events are dependent.

Multiplication Law

- Multiplication Law to Find Joint Probability

  For **any two events** rearranging the equation in the definition for conditional probability, we get:

- General Multiplication Rule:
  For any two events A and B,
  $P(A \cap B) = P(A)\ P(B|A)$ or
  $P(A \cap B) = P(B)\ P(A|B)$.

- For independent events
  $P(A \cap B) = P(A)\ P(B)$

- For more than two independent events
  $P(E_1 \cap E_2 \cap E_3 \ldots \cap E_k) = P(E_1)\ P(E_2)\ P(E_3) \ldots P(E_k)$

Confusion between Mutually Exclusive and Independent Events

- While mutually exclusive events cannot occur together, independent events must be able to occur together.

- Suppose that neither P(A) or P(B) = 0 and that A and B are mutually exclusive. Then
  $P(A \cap B) = 0$.

- If A and B are independent then
  $P(A \cap B) = P(A)\ P(B)$

- If A and B are mutually exclusive, they cannot be independent
  $P(A \cap B) = 0 \neq P(A)\ P(B)$
  since neither P(A) or P(B) = 0

Two Important Observations

- Two events are either mutually exclusive or independent.
  - Mutually exclusive events are always dependent.
  - Independent events are never mutually exclusive.
- Dependents events may or may not be mutually exclusive.

What can go Wrong?

- Beware of probabilities that don't add up to 1.
- Don't add probabilities of events if they're not disjoint.
- Don't multiply probabilities of events if they're not independent.
- Don't confuse disjoint and independent— disjoint events *can't* be independent.
- Whenever you see probabilities multiplied together, stop and ask whether you think they are really independent.

What can go Wrong?

- The addition rule for disjoint events can be generalized for any two events using the general addition rule.
- The multiplication rule for independent events can be generalized for any two events using the general multiplication rule.
- Conditional probabilities come from the general multiplication rule.
- Tree diagrams are helpful ways to display conditional events and probabilities.

What can go Wrong?

- Don't use a simple probability rule where a general rule is appropriate—don't assume that two events are independent or disjoint without checking that they are.
- Don't find probabilities for samples drawn without replacement as if they had been drawn with replacement.

Rolling One Die

- A = score on die is even = { }
- B = score on die is odd = { }
- C = score is greater than 4 = { }
- A∩B =
- AUB =
- A∩C =
- B∩C =
- (A∩C) U **(**B∩C) =
- (AUB) ∩C =

Example

Of 200 people interviewed regarding intention to buy your product and whether or not they are financially able to do so, the following results were obtained:

	To Buy (B)	Not to Buy ($\bar{B}$)	Total
Able to finance (A)	40	20	60
Unable to finance ($\bar{A}$)	80	60	140
Total	120	80	200

What is the probability that:

1. A customer has a desire to buy?
2. A customer is financially able and has a desire to buy?
3. A customer will buy given that she has the financial ability to pay?

Examples

Pull 3 cards from a deck with replacement.
What is the probability that all are black?
What is the probability that at least one is red?
What is the probability that all are different?

What is the probability that a 3 letter sequence will not have any repeated letters?

A box contains 100 chips, 75 perfect, 25 defective. Select 3 without replacement. What is the probability of all chips being in working order?

In a party of 4, what is the probability that at least 2 will have the same birthday?

Probability that their birthdays are different

n	p_n	n	p_n
2	0.9973	13	0.8056
3	0.9918	14	0.7769
4	0.9836	15	0.7471
5	0.9729	16	0.7164
6	0.9595	17	0.6850
7	0.9438	18	0.6531
8	0.9257	19	0.6209
9	0.9054	20	0.5886
10	0.8831	21	0.5563
11	0.8589	22	0.5243
12	0.8330	23	0.4927