EXPECTATIONS

The Mean of a Sample

Example 1: Salaries of 6 employees at a certain company (€000s) are:

$$20.3, 14.9, 18.9, 21.7, 16.3, 17.7$$

The average of the sample is

$$\overline{x} = \frac{20.3 + 14.9 + 18.9 + 21.7 + 16.3 + 17.7}{6} = 18.3$$

Generally, if $x_1, x_2, \cdots x_n$ in a sample of size n,

Then the sample average is

$$\overline{x} = \frac{x_1 + x_2 + \cdots x_n}{n}$$

It may be written as

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Example 2: Grouped Data

A sample of 116 teachers were measured for computer anxiety.

Category	**Score Range**	**Frequency**	**Relative Frequency**
Very relaxed	10 – 19	39	.33
Generally relaxed	20 – 26	24	.21
Some mild anxiety	27 – 32	23	.20
Anxious/tense	33 – 36	8	.07
Very anxious	37 – 50	22	.19

Source: Gordon H. R. D. (1995), " Analysis of Computer Anxiety Levels of Secondary School Teachers" *Journal of Studies in Technical Careers", 15, 2.*

The mean is obtained as follows:

$$\overline{x} = \frac{39 * 14.5 + 24 * 23 + 23 * 29.5 + 8 * 34.5 + 2 * 43.5}{116} = 26.11$$

Generally, for grouped data, the sample average is

$$\frac{f_1 x_1 + f_2 x_2 + \cdots f_k x_k}{f_1 + f_2 \cdots f_k}$$

where f_i and x_i are the frequencies and mid-point variable value respectively of the k groups. The mean of grouped data may be written as

$$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$$

Mean of a Random Variable

Definition: The mean of a discrete random variable is defined as the weighted average of all possible values. The weights are the probabilities of respective values of the random variable

$$E(X) = \sum_x xp(x)$$

The mean or expected value of a random variable X is often denoted by μ_x

Examples:

1. The probability density function (pdf) of the number of claims per policy holder in five years is as follows:

No. of Claims (X)	0	1	2	3	4	5	6
Probability (P(X=x))	.307	.206	.204	.114	.064	.078	.027

2. The future rates of a stock portfolio can be characterised by the following probability distribution:

Rate of Return	.20	.15	.10	.05	.00	-.05	-.10	-.15	-.20
Probability	.05	.15	.26	.20	.15	.10	.05	.03	.01

In each case calculate the expected value.

EXPECTED MONETARY VALUE

In a business environment it is usually more informative to express expectation in monetary terms.

Example

A coin is tossed ten times.

When a head is obtained €4 is won.

When a tail is obtained €2 is lost.

Decision Making using Expectation

Example 1

An investor has a certain amount of money to invest. Three alternative portfolio selections are available. The estimated profits depend on the economic conditions as follows:

	Profit (€000s)		
	Portfolio Selection		
	A	B	C
Economy declines	€0.5	€-2	€-7
No change	€1	€2	€-1
Economy expands	€2.5	€6	€20

The probability of occurrence of the economic conditions are:

P(economy declines) = 0.3

P(no change) = 0.5

P(economy expands) = 0.2

Determine the best portfolio for the investor.

Example 2

A potential customer for a €200,000 fire insurance policy has a home in an area that, according to experience, may sustain a total loss in a given year with a probability of .05 and a 50% loss with a probability of .03. There is a 92% chance that no claim will be made. Ignoring all other partial losses, what premium shoud the insurance company charge in order to break even?

VARIANCES

Variance of a Sample:

Spread of individual values from the mean.

Example 1:

Salaries of 6 employees in a certain company are: (£000)

$$20.3, 14.9, 18.9, 21.7, 16.3, 17.7$$

Recall $\overline{x} = 18.3$.

Calculation of s^2:

$$s^2 = \frac{\left[(20.3 - 18.3)^2 + (18.9 - 18.3)^2 + \ldots\right]}{5}$$

Generally, if $x_1, x_2, \cdots x_n$ is a sample of size n,

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

$$s^2 = \frac{\left[(20.3 - 18.3)^2 + (18.9 - 18.3)^2 + \ldots\right]}{5} = 6.368$$

Standard Deviation:

$$s = \sqrt{6.368} = 2.523$$

Example 2: Grouped Data

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$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2 f_i}{\sum_i f_i}$$

The variance is obtained as follows:

$$s^2 = \frac{39 * (14.5 - 26.11)^2 + 24 * (23 - 26.11)^2 + 23 * (29.5 - 26.11)^2 + \cdots}{115}$$

Check

$$s^2 = 112.78$$

Standard Deviation

$$s = \sqrt{s^2} = \sqrt{112.78} = 10.62$$

The variance of a discrete random variable

Definition: The variance is defined as the weighted average of the squared differences between each possible outcome and its mean ... the weights being the probability of the respective outcomes

$$V(X) = \sum_x (x - \mu_x)^2 p(x)$$

Or equivalently

$$V(X) = E\left(X - (E(X)\right)^2$$

Shortcut formula

$$V(X) = E(X^2) - (E(X))^2$$

The variance is often denoted by σ_x^2.

$$\sigma_x^2 \equiv E(X - \mu_x)^2$$

The standard deviation is denoted by σ_x.

$$\sigma_x \ = \sqrt{(E(X - \mu_x)^2)}$$

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Probability	.05	.15	.26	.20	.15	.10	.05	.03	.01

In each case calculate the variance.

PROPERTIES OF EXPECTATIONS

RECALL

$$E(X) = \sum_x xp(x) \text{ when } X \text{ is discrete}$$

Example 1

The average salary of employees in an advertising firm is €27,500. After negotiations with the trade union, it was agreed that employees would get a rise of €100 in addition to 10 percent increase on their basic salaries. What is the new average salary?

Let X = old salary; Y = new salary.

$$Y = 100 + 1.1X$$

If E(X) = €27,500, what is E(Y)?

Example 2

The sales per day for a product is estimated as follows based on past records.

Daily Sales (X)	Probability
20	.1
21	.3
22	.4
23	.2

(a) Obtain the mean and variance of the daily sales.

(b) If the profit can be described by the equation

$$Profit = -10 + 4X,$$

(i) What is the expected daily profit?

(ii) What is the variance of the daily profit?

Properties of Expected Values

1. E(X+c) = E(X) + c

$$
\begin{aligned}
E(X+c) &= \sum_x (x+c)p(x) \\
&= \sum_x xp(x) + c\sum_x p(x) \\
&= = E(X) + c.
\end{aligned}
$$

2. E(cX)= cE(X)

$$
\begin{aligned}
E(cX) &= \sum_x cxp(x) \\
&= c\sum_x xp(x) \\
&= cE(X).
\end{aligned}
$$

3. V(X+c) = V(X)

$$
\begin{aligned}
V(X+c) &= E[X+c-E(X+c)]^2 \\
&= E[X-E(X)]^2 \\
&= V(X)
\end{aligned}
$$

4. $V(cX) = c^2V(X)$

$$
\begin{aligned}
V(cX) &= E[cX-E(cX)]^2 \\
&= E[cX-cE(X)]^2 \\
&= E[c(X-E(X))]^2 \\
&= c^2E[X-E(X)]^2 \\
&= c^2V(X)
\end{aligned}
$$