POISSON DISTRIBUTION

Examples

1. Number of telephone calls in a week.
2. Number of people arriving at a checkout in a day.
3. Number of industrial accidents per month in a manufacturing plant.

Generally

X = number of events, distributed independently in time, occurring in a fixed time interval.

X is a Poisson variable with pdf:

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, \ldots, \infty$$

where λ is the average.

Example 1:

If the number of arrivals is 10 per hour on average, determine the probability that, in any hour there will be

(i) 0 arrivals;

(ii) 6 arrivals;

(iii) more than 6 arrivals.

Example 2:

The average rate of telephone calls in a busy reception is 4 per minute. If it can be assumed that the number of telephone calls per minute interval is Poisson distributed, calculate the probability that

(i) at least 2 telephone calls will be received in any minute.

(ii) any minute will be free of telephone calls.

(iii) no more than one telephone call will be received in any one minute interval.

Derivations of Some Properties of Poisson

1. Clearly

$$e^{-\lambda}\frac{\lambda^x}{x!} > 0 \ \text{ since } \ \lambda > 0$$

Also

$$\sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = 1$$

since

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

i.e.

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

2. $E(X) = \lambda$

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\
&= e^{-\lambda} \lambda \left[\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right] \\
&= e^{-\lambda} \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right] \\
&= e^{-\lambda} \lambda e^{\lambda} = \lambda
\end{aligned}
$$

APPLICATIONS OF THE POISSON

The Poisson distribution arises in two ways:

1. **As an approximation to the binomial when p is small and n is large**

   Example: In auditing when examining accounts for errors; n, the sample size, is usually large. p, the error rate, is usually small.

2. **Events distributed independently of one another in time:**

   X = the number of events occurring in a fixed time interval has a Poisson distribution.

   Example: X = the number of telephone calls in an hour.

$$PDF : p(x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, 1, 2, \cdots ; \lambda > 0$$

Poisson Approximation to the Binomial

When the sample size n is large and the occurrence p is small, the pdfs of the Binomial and Poisson are similar.

Example: An audit consists of inspecting 100 line items selected from a large accounting population. The accounts are rejected if more than three errors are found in the sample. Past experience indicates that, on average, 2% of the accounts have been misstated. Calculate the probability of rejecting the whole set.

x	Binomial $\binom{100}{x}.02^x.98^{100-x}$	Poisson Approximation $e^{-2}2^x/x!$
0	0.13262	0.13534
1	0.27065	0.27067
2	0.27341	0.27067
3	0.18228	0.18045
Total	0.85890	0.85713

Rule of Thumb: Use Poisson Approximation to Binomial when $n \geq 30$ and $p \leq .05$.

Poisson as an approximation to the binomial when n is large p is small

Recall:

- mean of binomial $= np$
- mean of Poisson $= \lambda$

PDF of Binomial

$$\begin{aligned} P(x) &= \binom{n}{x} p^x (1-p)^{n-x}; \quad p = \frac{\lambda}{n} \\ &= \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\ &= \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \frac{(1-\frac{\lambda}{n})^n}{\left(1-\frac{\lambda}{n}\right)^x} \rightarrow \frac{\lambda^x}{x!} e^{-\lambda} \end{aligned}$$

Now

$$\begin{aligned} (1 - \frac{\lambda}{n})^n &\rightarrow e^{-\lambda} \text{ as } n \rightarrow \infty \\ (1 - \frac{\lambda}{n})^x &\rightarrow 1 \text{ as } n \rightarrow \infty \\ \frac{n!}{(n-x)!n^x} &= \frac{n(n-1)\ldots(n-(x-1))}{n^x} \\ &= 1(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{x-1}{n}) \rightarrow 1 \end{aligned}$$

Example 1:

An auditor with a firm, running a chain of shops, believes that the number of errors in its invoices follows a Poisson distribution. To test the hypothesis, 100 shops were observed and the invoices were inpsected. The distribution of the number of errors found were as follows:

Errors	0	1	2	3	4	5
No. Shops	16	34	27	12	7	4

Example 2:

An auditor selects a simple random sample of line items to decide whether or not to audit the full set. If more than 10% of the sample are misstated, the entire set will be audited. In each of the following circumstances, estimate the probability of this happening when the population contains 5% misstated accounts:

1. A sample of 10 is selected from a set of 40;
2. A sample of 10 is selected from a set of 5,000;
3. A sample of 50 is selected from a set of 5,000.