Context-Free Languages: Answers

1.
    a. The following strings can be produced: aa, aab, aba, baa
    b. Four distinct derivations are as follows:

       S => AA => bAA => baA => babA => babbA => babbAb => babbab

       S => AA => AAb => Aab => Abab => Abbab => bAbbab => babbab

       S => AA => bAA => bAAb => bAbAb => bAbbAb => babbAb => babbab

       S => AA => AAb => bAAb => bAbAb => babAb => babbAb => babbab
2.
    a. The context-free grammar with start symbol S is as follows:

       S -> aSa
       S -> bSb
       S -> c
    b. The context-free grammar with start symbol S is as follows:

       S -> aSa
       S -> bSb
       S -> ε
    c. The context-free grammar with start symbol S is as follows:

       S -> aSa
       S -> bSb
       S -> a
       S -> b
       S -> ε
3.
    a. The derivation of the string ababba is as follows:

       S => aB => abS => abaB => ababS => ababbA => ababba
    b. In order to prove this, we need to jointly prove the following:

       All strings derived from S have the same number of a's and b's
       All strings derived from A have one more a than b
       All strings derived from B have one more b than a

       Base cases: the base cases are those production rules which have no non-terminals on the right-hand side. These are as follows:

       A -> a
       B -> b

       The required property follows immediately from these rules.

Inductive hypothesis: we assume that the required property is true for S, A and B.

Inductive cases: we show that the required property follows from the inductive hypothesis for those production rules which have non-terminals on the right-hand side

S -> aB

B generates strings which have one more b than a (by the inductive hypothesis). Therefore, strings generated from the right-hand side will contain the same number of a's and b's

S -> bA

A generates strings which have one more b than a (by the inductive hypothesis). Therefore, strings generated from the right-hand side will contain the same number of a's and b's

A -> aS

S generates strings which have the same number of a's and b's (by the inductive hypothesis).
Therefore, strings generated from the right-hand side will contain one more a than b

A -> BAA

A generates strings which have one more a than b (by the inductive hypothesis).
B generates strings which have one more b than a (by the inductive hypothesis).
Therefore, strings generated from the right-hand side will contain one more a than b

B -> bS

S generates strings which have the same number of a's and b's (by the inductive hypothesis).
Therefore, strings generated from the right-hand side will contain one more b than a

B -> ABB

A generates strings which have one more a than b (by the inductive hypothesis).
B generates strings which have one more b than a (by the inductive hypothesis).
Therefore, strings generated from the right-hand side will contain one more b than a

4. The parse tree is as follows:

```
          S
         /|\
        / | \
       P  V  P
      /|  |  |\
     / |  |  | \
    A  P ate A  P
   /   |     |   \
  /    |     |    \
big   Jim  green cheese
```

5. Two different parse trees are as follows:

```
   S               S
  / \             / \
 S   S           S   S
/|\ /|\          |  / \
(S) (S)          e S   S
  |   |            /|\ /|\
  e   e            (S) (S)
                    |   |
                    e   e
```

   An equivalent unambiguous grammar is as follows:

   S -> ST
   S -> ε
   T -> (S)

6.
   a. Yes
   b. No
   c. No
   d. No
   e. No
   f. Yes
   g. Yes
   h. Yes
   i. Yes
   j. No

7. The pushdown automaton with start state s and accepting state f has the following transitions:

   ((s,(,ε),(s,())
   ((s,),(),(s,ε))
   ((s,ε,ε),(f,ε))

8. The context-free grammar with start symbol S is as follows:

   S -> aSa
   S -> aSb
   S -> bSb
   S -> bSa
   S -> a

9. The pushdown automaton with start state s and accepting state f has the following transitions:

   ((s,ε,ε),(q1,ε))
   ((s,ε,ε),(q4,ε))
   ((q1,ε,ε),(q3,ε))
   ((q1,a,ε),(q1,a))
   ((q1,b,a),(q2,ε))

((q2,b,a),(q2,ε))
((q2,ε,ε),(q3,ε))
((q3,c,ε),(q3,ε))
((q3,ε,ε),(f,ε))
((q4,ε,ε),(f,ε))
((q4,a,ε),(q4,ε))
((q4,b,ε),(q5,b))
((q5,b,ε),(q5,b))
((q5,c,b),(q6,ε))
((q6,c,b),(q6,ε))
((q6,ε,ε),(f,ε))

10. In order to show that a language is not context-free using the pumping lemma, we must show that there are strings v, w, x, y and z such that |wy| > 0 and vw^kxy^kz is an element of the language for all k >= 0. We assume that the language is context-free and obtain a contradiction.

    a. Let s = $a^nb^{n+1}c^{n+2}$. s must equal vwxyz, where v, w, x, y and z satisfy the conditions above.

       If wy contains at least one a, then it can contain no c's. Therefore, vw^2xy^2z must contain at least n+1 a's and exactly n+2 c's, which is impossible if the string is in the language.

       If wy contains no a's, then it must contain either b or c. Therefore, vw^0xy^0z = vxz has either fewer than n+1 b's or fewer than n+2 c's, but in either case exactly n a's, which is impossible if the string is in the language.

    b. Let s = $a^nb^nc^n$. s must equal vwxyz, where v, w, x, y and z satisfy the conditions above.

       If wy contains at least one a, then it can contain no c's. Therefore, vw^0xy^0z = vxz must contain less than n a's, but exactly n c's, which is impossible if the string is in the language.

       If wy contains no a's, then vw^2xy^2z contains either more than n b's or more than n c's, but exactly n a's, which is impossible if the string is in the language.