Mathematical Prerequisites: Answers

1.
   a. True
   b. False
   c. True
   d. True
   e. True
   f. True
   g. False
   h. True
   i. False
2.
   a. {1,3,5,7}
   b. ∅
   c. ∅
   d. {{8},{7,8}.{8,9},{7,8,9}}
   e. {∅}
   f. {<1,1,1>,<1,1,2>,<1,1,3>,<1,2,1>,<1,2,2>,<1,2,3>}
   g. ∅
   h. {<∅,1>,<∅,2>,<{1},1>,<{1},2>,<{2},1>,<{2},2>,<{1,2},1>,<{1,2},2>}
3.
   a. This will be the case if g(f(A)) = C
   b. This will be the case if a ≠ b implies g(f(a)) ≠ g(f(b))
   c. This will be the case if both of the above conditions are satisfied.

   Note: It is possible for h to be surjective without f being surjective, and it is also possible for h to be injective without g being injective.

   For example: let A={a}, B={b,b'}, C={c} and f(a) = b, g(b) = c, g(b') = c

4.
   a. Because the three sets are countable, we can list their elements:

      First set: a_{00} a_{01} a_{02}...
      Second set: a_{10} a_{11} a_{12}...
      Third set: a_{20} a_{21} a_{22}...

      We can list all the elements of the union of these three sets (with possible repetitions) as follows:

      a_{00} a_{10} a_{20} a_{01} a_{11} a_{21} a_{02} a_{12} a_{22} ...

      Therefore, this set is countable. Since the union of the three sets is a subset of this, it must also be countable.

   b. The finite subsets of N may be arranged as follows:

      First row: all the singleton subsets of N, i.e., {1}, {2}, {3}, ...,
      Second row: all 2-element subsets of N, i.e., {1,2} , {1,3}, {2,3}, {1,4}, {2,4}, {3,4},

{1,5}, ...,
Third row: all 3-element subsets of N, i.e., {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}, {1,2,5}, {1,3,5}, {1,4,5}, {2,3,5}, {2,4,5}, {3,4,5}, etc.
and so on, enumerating all subsets of N by set size.

We may then enumerate the complete set using a dovetailing approach by visiting the sets in the corresponding array of sets in a diagonal fashion.

5.

a.
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f(n) = 2n+1
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b.
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f(n) = (n+1)/2, if n is odd
     = -n/2,    if n is even
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c. We define an ordering on triples as follows:

<x1,y1,z1> <x2,y2,z2>, if x1 + y1 + z1 < x2 + y2 + z2
<x1,y1,z1> <x2,y2,z2>, if x1 + y1 + z1 = x2 + y2 + z2 and x1 < x2
<x1,y1,z1> <x2,y2,z2>, if x1 + y1 + z1 = x2 + y2 + z2 and x1 = x2 and y1 < y2

This gives us the following ordering:

<0,0,0> <0,0,1> <0,1,0> <1,0,0> <0,0,2> <0,1,1> <0,2,0> <1,0,1> <1,1,0> <2,0,0> ...

We can now define a bijection f between N and this set as follows:

f(0) = (0,0,0), f(1) = (0,0,1), f(2) = (0,1,0), f(3) = (1,0,0), ...

6.

a. ∅ ∈ C (by 1)

Therefore {∅,∅} = {∅} ∈ C (by 2)

Therefore {∅,{∅}} ∈ C (by 2)

b. C does not contain any infinite sets because all its elements must be constructed by a finite number of applications of rules 1, 2 and 3. However, C does contain an infinite number of sets.

c. C is countable. We can list the elements of C as follows:

Apply rule 1, and then repeatedly apply rules 2 and 3 for all pairs of elements already listed.

7. Base Case (n=1)

RHS = $1 \times 2 \times 3 \times 4/4$

$= 1 \times 2 \times 3$

= LHS

Inductive Hypothesis: Assume true for n:

$1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$

Inductive Case: Show that it is true for n+1:

LHS = $1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + n(n+1)(n+2) + (n+1)(n+2)(n+3)$

$= n(n+1)(n+2)(n+3)/4 + (n+1)(n+2)(n+3)$ (by inductive hypothesis)

$= n(n+1)(n+2)(n+3)/4 + 4(n+1)(n+2)(n+3)/4$

$= (n+1)(n+2)(n+3)(n+4)/4$

= RHS

8. Base Case (n=0)

$n^3 + 2n = 0$ (which is divisible by 3)

Inductive Hypothesis: Assume true for n:

$n^3 + 2n$ is divisible by 3

Inductive Case: Show that it is true for n+1:

$(n+1)^3 + 2(n+1)$

$= n^3 + 3n^2 + 3n + 1 + 2n + 2$

$= (n^3 + 2n) + 3n^2 + 3n + 3$

$= (n^3 + 2n) + 3(n^2 + n + 1)$

Each of these terms is obviously divisible by 3 (the first by the inductive hypothesis)

Therefore the predicate is true for all n

9. The sets {...,-3,-2,-1,0} and {0,1,2,3,...} are both countably infinite. Their intersection {0} is finite.

The sets Z (integers) and Z+ (positive integers) are both countably infinite. Their intersection (Z+) is countably infinite.

The sets of real numbers in the intervals [-1,0] and [0,1] are both uncountable. Their intersection {0} is finite.

The sets R+ ∪ N (positive reals and naturals), and R- ∪ N (negative reals and naturals) are both uncountable. Their intersection (N) is countably infinite.

The sets R (reals) and R+ (positive reals) are both uncountable. Their intersection (R+) is uncountable.

10. Assume we have an uncountable set S and a countable set T.

S ∩ T is countable, since it is a subset of T.

If S-T were countable, then (S-T) ∪ (S ∩ T) = S would be also.