The Leslie Matrix

- The *Leslie* matrix is a generalization of the above.
- It describes annual increases in various age categories of a population.
- As above we write \( p_{n+1} = Ap_n \) where \( p_n, A \) are given by:

\[
p_n = \begin{pmatrix} p^n_1 \\ p^n_2 \\ \vdots \\ p^n_m \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \ldots & 0 & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \sigma_{m-1} & 0 \end{pmatrix}
\]

(3.42)

\( \alpha_i, \sigma_i \), the number of births in age class \( i \) in year \( n \) & probability that \( i \) year-olds survive to \( i + 1 \) years old, respectively.

---

The Leslie Matrix (/2)

- Long-term population demographics found as with Eqn.(3.21) using \( \lambda \)s of \( A \) in Eqn.(3.42) & \( \det(A - \lambda I) = 0 \) to give Leslie characteristic equation:

\[
\lambda^n - \alpha_1 \lambda^{n-1} - \alpha_2 \sigma_1 \lambda^{n-2} - \alpha_3 \sigma_1 \sigma_2 \lambda^{n-3} - \cdots - \alpha_n \prod_{i=1}^{n-1} \sigma_i = 0
\]

(3.43)

\( \alpha_i, \sigma_i \), are births in age class \( i \) in year \( n \) & the fraction that \( i \) year-olds live to \( i + 1 \) years old, respectively.
Eqn.(3.43) has one +ive eigenvalue $\lambda^*$ & corresponding eigenvector, $v^*$.

For a general solution like Eqn.(3.19)

$$P_n = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + \cdots + c_m \lambda_m^n v_m, \quad (3.44)$$

with dominant eigenvalue $\lambda_1 = \lambda^*$ gives long-term solution:

$$P_n \approx c_1 \lambda_1^n v_1 \quad (3.44)$$

with stable age distribution $v_1 = v^*$. The relative magnitudes of its elements give stable state proportions.

---

Example 3.4: Leslie Matrix for a Salmon Population

- Salmon have 3 age classes & females in the 2nd & 3rd produce 4 & 3 offspring, each season.
- Suppose 50% of females in 1st age class survive to 2nd age class & 25% of females in 2nd age class live on into 3rd.
- The Leslie Matrix (c.f. Eqn.(3.43)) for this population is:

$$A = \begin{pmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix} \quad (3.45)$$

- Fig. 3.4 shows the growth of age classes in the population.
Example 3.4: Leslie Matrix for a Salmon Population

The eigenvalues of the Leslie matrix may be shown to be

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & -1.309 & 0 \\ 0 & 0 & -0.191 \end{pmatrix} \quad (3.46)$$

and the eigenvector matrix $S$ to be given by

$$S = \begin{pmatrix} 0.9474 & 0.9320 & 0.2259 \\ 0.3158 & -0.356 & -0.591 \\ 0.0526 & 0.0680 & 0.7741 \end{pmatrix} \quad (3.47)$$

- Dominant e-vector: $(0.9474, 0.3158, 0.0526)^T$, can be normalized (divide by sum), to $(0.72, 0.24, 0.04)^T$. 

\[\]
Example 3.4: Leslie Matrix for a Salmon Population cont’d
- Long-term, 72% of pop’n are in 1st age class, 24% in 2nd and 4% in 3rd.
- Thus, due to principal e-value $\lambda_1 = 1.5$, population increases.
- Can verify by taking any initial age distribution & multiplying it by $A$.
- It always converges to the proportions above.

A side note on matrices similar to the Leslie matrix.

Any lower diagonal matrix of the form

$$
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
$$

(3.48)

Can ‘move’ a vector of age classes forward by 1 generation e.g.

$$
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}
$$

(3.49)
Stability in Difference Equations

- If difference equation system has the form \( u_n = A u_{n-1} \), then growth as \( n \to \infty \) depends on the \( \lambda_i \) thus:
  - If all eigenvalues \( |\lambda_i| < 1 \), system is stable & \( u_n \to 0 \) as \( n \to \infty \).
  - Whenever all values satisfy \( |\lambda_i| \leq 1 \), system is neutrally stable & \( u_n \) is bounded as \( n \to \infty \).
  - Whenever at least one value satisfies \( |\lambda_i| > 1 \), system is unstable & \( u_n \) is unbounded as \( n \to \infty \).

Markov Processes

- Often with difference equations don’t have certainties of events, but probabilities.
- So with Leslie Matrix Eqn.(3.42):
  \[
  p_n = \begin{pmatrix}
  p_{n1} \\
  p_{n2} \\
  \vdots \\
  p_{nm}
  \end{pmatrix}, \quad
  A = \begin{pmatrix}
  \alpha_1 & \alpha_2 & \ldots & \alpha_{m-1} & \alpha_m \\
  \sigma_1 & 0 & \ldots & 0 & 0 \\
  0 & \sigma_2 & \ldots & \ldots & \ldots \\
  0 & 0 & \ldots & \sigma_{m-1} & 0
  \end{pmatrix}
  \] (3.50)

\( \sigma_i \) is probability that \( i \) year-olds survive to \( i+1 \) year olds.

- Leslie model resembles a discrete-time Markov chain
- Markov chain: discrete random process with Markov property
- Markov property: state at \( t_{n+1} \) depends only on that at \( t_n \).

- The difference between Leslie model & Markov model, is:
  - In Markov \( \alpha_m + \sigma_m = 1 \) for each \( m \).
  - Leslie model may have these sums \(<1\).
Markov Processes (/2)

Stochastic Processes

- A Markov Process is a particular case of a Stochastic\(^8\) Process.
- A Markov Process is a Stochastic Process where probability to enter a state depends only on last state & on governing Transition matrix.
- If Transition Matrix has terms constant between subsequent timesteps, process is Stationary.

\[ \text{Markov Chain} \]

**Figure 3.5:** General Case of a Markov Process © Max Heimel, TÜ Berlin

\(^8\)One where probabilities govern entering a state

Markov Processes (/3)

- General form of discrete-time Markov chain is given by:

\[ u_{n+1} = Mu_n \]

where \( u_n, M \) are given by:

\[ u_n = \begin{pmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{pn} \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1p} \\ m_{21} & m_{22} & \cdots & m_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p1} & m_{p2} & \cdots & m_{pp} \end{pmatrix} \]  \hspace{1cm} (3.51)

- \( M \) is \( p \times p \) Transition matrix & its \( m_{ij} \) terms are called Transition probabilities such that \( \sum_{j=1}^p m_{ij} = 1 \).
- \( m_{ij} \) is probability that that item goes from state \( i \) at \( t_n \) to state \( j \) at \( t_{n+1} \).
Example 3.5: Two Tree Forest Ecosystem

- In a forest there are only two kinds of trees: oaks and cedars.
- At any time $n$ sample space of possible outcomes is $(O, C)$
- Here $O = \%$ of tree population that is oak in a particular year and $C = \%$ that is cedar.
- If same life spans & on death same chance an oak is replaced by an oak or a cedar
- But that cedars are more likely ($p = 0.74$) to be replaced by an oak than another cedar ($p = 0.26$).
- How can we track changes in the different tree types with time?

This is a Markov Process as oak/cedar fractions at $t_{n+1}$ etc are defined by those at $t_n$.

Transition Matrix (from Eqn.(3.51)) is Table 3.1:

<table>
<thead>
<tr>
<th></th>
<th>Oak</th>
<th>Cedar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oak</td>
<td>0.5</td>
<td>0.74</td>
</tr>
<tr>
<td>Cedar</td>
<td>0.5</td>
<td>0.26</td>
</tr>
</tbody>
</table>

**Table 3.1: Tree Transition Matrix**

Table 3.1 in matrix form is:

$$M = \begin{pmatrix} 0.5 & 0.74 \\ 0.5 & 0.26 \end{pmatrix}$$ (3.52)
Example 3.5: Two Tree Forest Ecosystem

- To track system changes, let $u_n = (o_n, c_n)^T$ be probability of oak & cedar after $n$ generations.
- If forest is initially 50% oak and 50% cedar, then $u_0 = (0.5, 0.5)^T$.
  Hence
  $$u_n = Mu_{n-1} = M^n u_0$$ (3.53)
- $M$ can be shown to have one positive $\lambda$ & corresponding eigenvector $(0.597, 0.403)^T$.
- This is the distribution of oaks and cedars in the $n$th generation.

Example 3.6: Soft Drink Market Share

- In a soft drinks market there are two Brands: Coke & Pepsi.
- At any time $n$ sample space of possible outcomes is $(P, C)$.
- Here $P =$ % market share that is Pepsi's in one year and $C =$ % that is Coke's.
- Know that chance of switching from Coke to Pepsi is 0.1.
- And the chances of someone switching from Pepsi to Coke are 0.3.
- How can the changes in the different proportions be modelled?
Example 3.6: Soft Drink Market Share

- This is a Markov Process as shares of Coke/Pepsi at \( t_{n+1} \) are defined by those at \( t_n \).
- Transition Matrix (from Eqn.(3.51)) is Table 3.2:

<table>
<thead>
<tr>
<th></th>
<th>Coke</th>
<th>Pepsi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coke</td>
<td>0.9</td>
<td>0.3</td>
</tr>
<tr>
<td>Pepsi</td>
<td>0.1</td>
<td>0.7</td>
</tr>
</tbody>
</table>

**TABLE 3.2:** Soft Drink Market Share Matrix

- Table 3.2 in matrix form:

\[
M = \begin{pmatrix}
0.9 & 0.3 \\
0.1 & 0.7 \\
\end{pmatrix}
\]  

(3.54)

The eigenvalues of the matrix in Eqn(3.53) are 1, \( \frac{3}{5} \).

The largest eigenvector, can be found to be \((0.75, 0.25)^T\).

This is the proportions of Coke and Pepsi in the \( n \)th generation.
Absorbing States
A state of a Markov Process is said to be absorbing or Trapping if $M_{ii} = 1$ and $M_{ij} = 0 \forall j$.

Absorbing Markov Chain
A Markov Chain is absorbing if it has one or more absorbing states. If it has one absorbing state (for instance state $i$), then the steady state is given by the eigenvector $X$ where $X_i = 1$ and $X_j = 0 \forall j \neq i$.

Example 3.6: Soft Drink Market Share, Revisited

- As Soft Drinks market is ‘liquid’, KulKola decides to trial product Brand ‘X’.
- Despite its name, Brand ‘X’ has potential to ‘Shift the Paradigm’ in Cola consumption.
- They think, inside 5 years, they can capture nearly all the market.
- Investigate if this is true, given that they take 20% of Coke’s share and 30% of Pepsi’s per annum.

*from KulKola’s Marketing viewpoint*
Example 3.6: Soft Drink Market Share

- Again, shares of Coke/Pepsi/Brand ‘X’ at \( n + 1 \) etc are defined by those at \( n \).
- Transition Matrix (from Eqn.(3.51)) is Table 3.3:

<table>
<thead>
<tr>
<th></th>
<th>Coke</th>
<th>Pepsi</th>
<th>Brand ‘X’</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coke</td>
<td>0.6</td>
<td>0.4</td>
<td>0</td>
</tr>
<tr>
<td>To Pepsi</td>
<td>0.2</td>
<td>0.3</td>
<td>0</td>
</tr>
<tr>
<td>Brand ‘X’</td>
<td>0.2</td>
<td>0.3</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 3.3:** Soft Drink Market Share Matrix Revisited

- Table 3.3 in matrix form:

\[
M = \begin{pmatrix}
0.6 & 0.4 & 0 \\
0.2 & 0.3 & 0 \\
0.2 & 0.3 & 1
\end{pmatrix}
\]

(3.55)

\( \lambda_{\text{max}} \) of the matrix in Eqn(3.55) is 1.
- \( \mathbf{v}_{\text{max}} \) is \((0, 0, 1)^T\) giving the shares of Coke, Pepsi and Brand ‘X’ in the \( n \)th generation, respectively.
Markov Processes (/13): Hidden Markov Models

- Markov Models have a visible state
- So transition probabilities & matrix are observable.
- In Hidden Markov Models visibility restriction is relaxed
- The transition probabilities are generally not known.
- Possibly observer sees underlying variable thro noise layer.

![Figure of Markov Process](image)

**Figure 3.8:** Hidden Markov Process © Max Heimel, TÜ Berlin

Applications of Non-Linear Models: Logistic Growth

- Linear difference equations are useful as permit closed-form solutions to be easily obtained.
- However, solutions often have don’t agree with observation.
- In many areas of science & esp. population biology, non-linear models are better (i.e. more realistic).
- Here look at simple non-linear models for population growth over time.
- The simplest model is the logistic equation
- This can have stability problems but is a very useful, basic model
- Look at logistic equation in discrete & later continuous form.